XIII. General Relativity.

XIII.1. Astrophysics and hydrodynamics

The original impetus for developing an action principle for hydrodynamics and thermodynamics was to prepare for a study of astrophysics, the structure of stars and galaxies. The relation of these fields of science to each other is less than evident. However, already in 1870 the physicist Homer Lane published a paper with the following title:

"On the Theoretical Temperature of the Sun, under the Hypothesis of a gaseous Mass maintaining its Volume by its internal Heat and depending on the laws of gases as known to Terrestrial Experiments" (Lane 1870)

Half a century later Eddington marveled at the success of an approach that treated the interior of the Sun as if it consists of an ideal gas. And even today the best that can be done to describe the interior structure of stars is to apply the experience that has been gained by terrestrial experiments.

We are going to consider the hydrodynamics of self gravitating fluids by means of Einstein's equations for the metric, in the context of the General theory of Relativity. Stars and galaxies tend to rotate and in some it may be said that rotation is their dominant characteristic; this has been the principal difficulty for some time. It is a problem that the preceding chapter has already prepared us for, but more preparation is needed: we can approach General Relativity only after promoting our theory to a relativistic, Lagrangian field theory in the sense of the Special Theory of Relativity. This is what we are going to try to do.

This sketch of an introduction to General Relativity divides naturally into several parts:

A very short summary of the first stage of General Relativity. The elevation of the Lorentzian metric of Special Relativity to a dynamical field with matter as its source. The development of a dynamical action principle for the metric. The Bianchi identity. The search for a hydrodynamical source for the metric field. The phenomenological approach. Gravity waves. Action principle.

XIII.2. Introduction to General Relativity. The metric.

Special relativity was created in 1905. It arose from the demand that the laws of mechanics have the same invariance group as Maxwell's theory of electromagnetism, the Poincaré group or the inhomogeneous Lorentz group. It became necessary to formulate physical laws in a language that makes Lorentz invariance manifest, in order that the attention could be directed to physics without the need to worry about consistency with the new principles at each stage. This had the effect of introducing the Lorentzian metric into all the equations of fundamental physics. For example, Maxwell's field equations now take the form

$$g^{\mu\nu}\partial_\mu F_{\nu\lambda} = J_\lambda. \tag{13.2.1}$$

The prominence of the metric led Einstein to seek a larger role for it, and this resulted in the idea of geodesic motion. The next great leap forward was the realization that Newton's equations for the motion of a particle in a gravitational field could be interpreted as geodesic motion in the special metric

$$ds^2 = (1 - \frac{2\varphi}{c^2})(cdt)^2 - d\vec{x}^2, \tag{13.2.2}$$

where φ is the Newtonian gravitational potential. The distance between two points in space time is

$$\int_a^b ds = \int_a^b \sqrt{-g}\sqrt{g_{\mu\nu}dx^\mu dx^\nu} = \int_a^b \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\tau, \quad \dot{x}^\mu := dx^\mu/d\tau, \tag{13.2.3}$$

and the equation that minimizes the value of this expression is [1]

$$\ddot{x}^\mu - \Gamma^\mu_{\nu\lambda}{}^\mu\dot{x}^\nu\dot{x}^\lambda = 0, \tag{13.2.4}$$

[1] This is an oversimplification. The formula (13.2.4) is valid if the parameter τ is chosen so as to make $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$.

where the dots stand for derivatives with respect to s and Γ is the metric connection,

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}(g_{\nu\rho,\lambda} + g_{\rho\lambda,\nu} - g_{\rho\lambda,\rho}). \tag{13.2.5}$$

The proliferation of indices is frightening, but most of the coefficients of Γ are zero and a little work leads to the revelation that Eq.(13.2.4), in the non relativistic limit with (13.2.2) is precisely the Newtonian equations of motion for a particle with unit mass in the field φ. This 'geometric' interpretation of gravity has had a great appeal, but we shall take the attitude, by now prevalent, that the metric is a dynamical field, analogous to the Maxwell potential. As we shall see, this was Einstein's standpoint as well.

XIII.3. Introduction to G.R. Dynamical metric

The recognition of geodesic motion was not just an interpretation of Newton's equations of motion; it was a major discovery with important experimental consequences. But the theory needed further development. In Newtonian theory the potential was determined by solving the Poisson equation with the mass distribution as a source,

$$\Delta\phi = -4\varpi G\rho, \tag{13.3.1}$$

with G a universal constant. It relates the field φ to the source ρ, the matter density. The search for a suitable generalization would take 10 years of labor by Einstein, Hilbert, Poincaré and others.

The answer was found by insisting (1) that the main equations of the theory must have the same form in all space time coordinate systems and (2) that the theory must be governed by an action principle. Here is how it was done.

In practical terms it is obvious that equations that look the same in all coordinate systems must be relations among tensor fields. Besides, for a physicist it is clear that the field components $g_{\mu\nu}$ must satisfy differential equations. The difficulty is that the derivatives of the metric do not form a tensor.

However, a 'covariant derivative' is defined with the help of a connection,

$$D_\mu := \partial_\mu - \Gamma_\mu;$$

the covariant derivative of a tensor is a tensor. This merely tells us that a connection is needed, but the remarks about geodesics strongly suggests that the metric connection (13.2.4) is to be used. This makes the metric tensor 'covariantly constant',

$$D_\mu g_{\nu\lambda} = 0.$$

It is the most economical choice and it simplifies everyhing.

The Gaussian curveture tensor is

$$[D_\mu, D_\nu] = R_{\mu\nu} = R_{\mu\nu}{}^b_a L^a_b,$$

where the (L^b_a) are matrices, the generators of 'local Lorentz transformations'. Alternatively, if A is any co-vector field,

$$[D_\mu, D_\nu]A_\rho = (R_{\mu\nu})^\lambda_\rho A_\lambda.$$

Two contractions leads to the curvature scalar $R = g^{\mu\rho}(R_{\mu\nu})^\nu_\rho$ and this scalar field, by virtue of being the only candidate with required characteristics, is the Lagrangian density for the metric field.

Variations of the action

$$A_{\rm metric} = \int d^4x\sqrt{-g}R \tag{13.3.2}$$

with respect to the metric,

$$\delta A_{\rm metric} = \frac{1}{2}\int d^4x\sqrt{-g}\,\delta g^{\mu\nu}G_{\mu\nu} = 0. \tag{13.3.3}$$

defines the Einstein tensor

$$G_{\mu\nu} = 2\frac{\delta R}{\delta g^{\mu\nu}} - g_{\mu\nu}R.$$

and gives a unique field equation

$$G_{\mu\nu} = 0. \quad \textit{Einstein's equation in vacuo} \tag{13.3.4}$$

It is a set of second order differential equations for the components of the metric, an elaborate generalization of Eq. (13.3.1). It makes no reference to any other fields or sources; it characterizes the metric field in a space time that is empty (we shall refer to it as a 'vacuum') except for the metric itself.

The discovery of field equations for the gravitational metric in empty space was a milestone in the development of a General Theory of Gravitation. It has received a direct confirmation only recently, with the discovery of traveling gravitational waves. But it is not yet a theory of gravitation. Just as the mass density ρ appears as a source for the potential in Newton's theory, we need to add sources to the right hand side of (13.3.4). This hydrodynamical source has remained unknown for 100 years.

We shall take great pains to justify this last statement. The difficulty is related to the Bianchi identity.

XIII.4. The Bianchi identity

Theorem. The Einstein tensor, defined by (13.3.3), satisfies the following equation,

$$D_\nu G_\mu{}^\nu = 0, \qquad \textit{Bianchi identity} \tag{13.4.1}$$

identically.

Proof. The action (13.3.2) is invariant under infinitesimal coordinate transformations,

$$\delta x^\mu = \xi^\mu,$$

under which

$$\delta g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu.$$

Hence

$$\delta A_{\rm metric} = 2\int d^4x\sqrt{-g}(D_\mu\xi_\nu)G^{\mu\nu} \tag{13.4.2}$$

is identically zero. An integration by parts gives

$$\int d^4x\sqrt{-g}`\, \xi_\nu D_\mu G^{\mu\nu} = 0.$$

Since the vector field ξ is arbitrary this implies that the statement (13.4.1) is true.

Implications of the Bianchi identity

Let us add a source to Einstein's equation (13.3.4):

$$G_{\mu\nu} = T_{\mu\nu}. \qquad Einstein's\ Equation\ cum\ fons. \qquad (!3.4.3)$$

Applying the covariant derivative we find

$$D_\nu G^{\mu\nu} = D_\nu T^{\mu\nu} = 0. \tag{13.4.4}$$

The left side is identically zero; therefore an inevitable consequence is that **there can be no solution of Einstein's equation for any source unless**

$$D_\mu T^{\mu\nu} = 0. \quad \textit{The Bianchi constraint} \tag{13.4.5}$$

Eq(13.4.2) is a condition of integrability. This condition was received with great enthusiasm and led to the first examples of interactive gravitation theory, among the most important a theory of the metric tensor field interacting with the vector field of electromagnetism, Einstein-Maxwell theory. We shall see how it came about.

This is where Emmy Noether entered history. We have seen, in Sections III.8 that, if an action is constructed from scalar fields $\phi, \psi, ...$ alone then there is a tensor field

$$T_\mu{}^\nu = \sum_\phi \phi_\mu \frac{\partial \mathcal{L}}{\partial \phi_\nu} - \delta_\mu^\nu \mathcal{L} \tag{13.4.6}$$

that satisfies a 'conservation law that resembles (13.4.2), with ∂_μ instead of D_μ..

Theorem. (Noether). Consider a theory of scalar fields defined by an action

$$A_{\rm matter}[g, \phi, \psi, ...] = \int d^4x \sqrt{-g}\mathcal{L},$$

in which the tensor $g = (g_{\mu\nu})$ is a Riemannian metric tensor. Suppose further that the action is invariant under infinitesimal coordinate transformations; that is, under infinitesimal variations with

$$\delta g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu, \qquad \delta\phi = \xi\phi, ... \tag{13.4.7}$$

Finally, suppose that the fields $\phi, \psi, ...$ satisfy the Euler-Lagrange equations of the action $A_{\rm matter}$.. Then the tensor field with components

$$T_\mu{}^\nu = \sum_\phi \phi_\mu \frac{\partial \mathcal{L}}{\partial \phi_\mu} - \delta_\mu^\nu \mathcal{L}$$

satisfies the divergence condition (13.4.5). Furthermore, the tensor T defined here is the same as the tensor

$$T^{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - g^{\mu\nu}\mathcal{L}. \tag{13.4.8}$$

Sketch of proof. The proof of Noether's second theorem in Section X.8 applies with some changes. The components of the metric may depend on the coordinates, but this can be allowed for by interpreting the variations (13.4.4) of the fields in terms of covariant derivatives. Partial integration then brings in the covariant derivative of the metric that, as we have seen, vanishes. The essential points are that the covariant derivative acts as a derivation of the tensor algebra and that it allows to carry through the partial integrations as usual.

We omit details of the proof but verify the statement in the special case of the relativistic action associated with Lagrange's theory, when

$$A_{\rm matter} = \int d^4x\sqrt{-g}\left(\frac{\rho}{2}(g^{\mu\nu}\psi_\mu\psi_\nu - c^2) - f - sT\right). \tag{13.4.9}$$

In this case

$$2\frac{\partial\sqrt{-g}A_{\rm matter}}{\partial g_{\mu\nu}} = \rho\psi_\mu\psi_\nu - g_{\mu\nu}\mathcal{L}, \tag{13.4.10}$$

which is the Noetherian energy momentum tensor. In the general case of the complete, conservative hydrodynamics the calulation is more complicated due to the appearance of tensor field variables.

This result points to a theory that includes Einstein's metric and Maxwell's electromagnetic fields with mutual interaction. Later the inclusion of Dirac's electronic field was worked out by the same principles. The question that inevitably arose was this? How can we add hydrodynamics to this family of interacting theories? A preliminary answer was soon found.

Particles as sources. Gravity waves

Formulating an action principle for a metric in interaction with a number of particles has been attempted in connection with the recent observation of Gravity Waves (LIGO). A finite number of point particles, intended as models of "Black Holes", are moving in a metric field. The matter action is a sum of contributions, one from each particle,

$$A = \int d^4x\sqrt{-g}\mathcal{L}_{particles} = \sum_a\int d\tau_a\int d^4x\sqrt{-g}\sqrt{g_{\mu\nu}\dot x_a^\mu(\tau_a)\dot x_a^\nu(\tau_a)}\delta^4(x-x_a).$$

Each term is the familiar geodesic action.. the inclusion of the factor $\sqrt{-g}$ may be unexpected. The dynamical variables are the four-dimensional positions ${x_a}^\mu$ of the objects, following paths in 4-space, each parameterized by its own "proper time" τ_a. So far, when this action is considered on its own, the metric is fixed, not yet a dynamical variable.

We have two ways to obtain the energy momentum tensor. The result is, of course, exactly the same. If we make the usual choice of the parameters, the proper times, then each of the big square roots is equal to 1 and Noether's formula gives

$$T^{\mu\nu} = \sum_a \left(\dot{x}_a^\nu \dot{x}_a^\mu - \frac{1}{2} g^{\mu\nu} \mathcal{L}_{\rm particles}\right) \delta^4(x - x_a), \tag{13.4.11}$$

This is very similar to (13.4.7). The second term comes from the factor $\sqrt{-g}$. The same result is obtained by variation of the metric

XIII.5. Sources for G.R., Tolman's formula

Following the main ideas a source for hydrodynamics is provided by the action (13.4.9); it was proposed some time ago and it was used for a preliminary study of the internal constitution of stationary stars (Fronsdal 2007).

But to stay closer to the historical sequence we return to 1934. By this time it must have been clear that the success of Einstein-Maxwell theory would have little impact on the problem of the effect of gravity on extended mass distributions. No action principle that could serve as a basis for relativistic hydrodynamics had been found and in 1934 Richard Tolman proposed, *faute de mieux*, a phenomenological formula for the right hand side of Einstein's equation,

$$T_{\mu\nu} = (\rho + p/c^2) U_\mu U_\nu - g_{\mu\nu}.$$

It was based on Tolman's review of Cauchy's approach to hydrodynamics. [2] The vector field U was interpreted as 4-velocity, reflecting the usual practice of upgrading all 3-dimensional velocity fields to 4-vectors. To agree with experience in particle theory the length was restricted,

$$g_{\mu\nu} U^\mu U^\nu = c^2,$$

[2]The term p/c^2 appears to be an error, made possible because Tolman followed the practice of setting the speed of light equal to unity. The combination $\rho + p/c^2$ is unnatural and certainly not justifiable in a formula that was wholly inspired by non relativistic physics; the natural combination $h + p$, where h is the hamiltonian density, is the enthalpy density. The confusion between energy density and mass density and ensuing conflicts are evident in a famous paper by Oppenheimer and Volkov (1949), see footnote on page

imposed as a constraint. The effect is to make this theory incompatible with the equation of continuity.

But this critique of Tolman's formula is actually irrelevant; What makes us determined to eschew Tolman's approach is that it fails to take into account the Bianchi constraint.

The question still remained: what is the source that would define relativistic hydrodynamics?

XIII. 6. Conservative sources for Relativy

We have already stressed the importance of action principles in physics. In the context under discussion an action principle has been found that gives a highly satisfactory answer to a very important problem: to find a source for Einstein's equation that respects the integrability condition (13.4.2).

Unfortunately, for more than 80 years a consensus has been maintained, to the effect that the problem has no solution. For a large part of this period countless papers have been published that have acted upon Tolman's suggestion. In the early part of this period there were those who hoped for, and believed in, a better solution to the problem. Today, the idea has lost all support, being almost universally regarded as impossible. But this is a defeatist position and it leaves General Relativity in a deplorable state.

What is it that we are trying to do? We have in hand a theory of the metric field and several theories of matter, more specifically hydrodynamics. We wish to couple them to each other. That is our problem, but many do not see it that way. The attitude that seems to prevail is that Einstein's equation is a point of departure, subject to be developed and improved by adding additional terms to it. Note that the 'variables' ρ, p and U are just letters, with no *a priori* properties, they are not the variables of a theory or model. All that we can know about them is the information that comes from the gravitational field equation $G = T$ itself. This information is very unlikely to constitute a theory.

To demonstrate this suppose that we have a theory that passes for hydrodynamics, a theory with variables ρ, p and U, and interpret the corresponding letters in Tolman's formula accordingly. If the Bianchi constraint is not satisfied on shell then the gravitational field equation selects a subset of the solutions of that theory. We know from experience, most clearly from quantum theory, that this is likely to make the theory untenable. In quantum mechanics it leads to a violation of unitarity; in our present situation to a loss of mass conservation.

To end this protest against the *status quo* let us raise a point that is the most compelling reason for persisting in respecting the imperative conditions of integrability; it is this: when we compromise we loose the hope of making predictions. It is often pointed out, in defense of Tolman's formula, that it can easily be generalized, and when carefully considered this is the strongest objection that can be raised against it.

XIV. Superfluid Helium-4. Part B.

This chapter remains to be written. Here is an preview of what may appear in a second edition of this book.

Two fluids or two flows

Landau, in his first, pathbreaking paper, strongly emphasized that we should not think of Helium below the λ point as a mixture of two kinds of fluids; "there is only one kind of Helium atom". Instead, there are 2 kinds of 'flow'. Nevertheless, Tisza proposed his 2-fluid theory, and this point of view has since become dominant. Yet Tilley and Tilley (1974, 1986) (and others) make it a point to say that the two-fluid hypothesis should not be taken literally.

In the first experiments Helium below the λ point flowed from vessel A to vessel B through a very narrow channel, apparently without resistance and without transporting heat. This suggested to London that the transported fluid has zero entropy and to Tisza that the flow consists of only one of two component that make up the fluid, a superfluid with zero entropy. But the fluid that accumulates in vessel B is not pure superfluid and in fact pure superfluid exists only at absolute zero of temperature. Other experiments have confirmed that the concentration (or what is interpreted as concentration) is completely determined by temperature and pressure. The 'concentration' is not an independent dynamical variable and the idea of a mixture has no sense. Measurements of concentration are indirect and other interpretations are possible.

One set of experiments of this kind, by Andronikashvili (1946) and others, has a stack of parallel discs suspended by a wire, rotating as a torsion pendulum. The interpretation depends on determining the angular momentum and the period. We have seen, in Section X. .. that the former is defined in a conservative context. The calculation that was done to justify a conclusion about the concentration is guess that cannot be justified.

In a similar experiment a cylindrical vessel was filled with Helium and rotated at constant angular speed. The temperature was lowered below the λ point and the cylinder was brought to rest. After some time the fluid appeared to have come to rest as well. Then, after a delay of as much as 2 weeks the temperature was brought back up above the λ point and the fluid started to flow, demonstrating that the system remembered some of its original angular momentum. The accepted interpretation is that, at the lower temperature the normal fluid was at rest while the superfluid moved through it without encountering any resistance. A more attractive hypothesis, possible in a fluid subject to Conservative Hydrodynamics, is that the flow velocity $\vec{v}$ is zero while the angular momentum is not. This would make the angular momentum invisible.

Perhaps it may be permitted to ask whether conservative hydrodynamics may offer some hope of an alternate understanding of the odd behavior of superfluids. It is commonly held that what makes this fluid different is that it remains liquid at very low temperatures. That is to say that, except for this accident, other fluids would behave the same way. That might have made it possible, on the basis of a more advanced form of hydrodynamics, to predict the odd behavior of liquid Helium.