XII. Special relativity

Einstein's theory of General Relativity is a theory of a dynamical metric field interacting with particles. This book exists because there is (or was) no satisfactory theory of the dynamical metric interacting with extended distributions of matter. It will be shown in Chapter 13 that any theory of interacting fields that includes the Einsteinian metric must be based on an action principle.

The non relativistic limit of such a theory, if it can be found, is a form of hydrodynamics, necessarily formulated as an action principle. One approach to the problem is therefore to study action principles for hydrodynamics, then step up, first, to a field theory that is Lorentz invariant and thereafter to one that is invariant under all coordinate transformations. We already have the theory of Lagrange, discovered by Lagrange and Fetter and Walecka. The relativistic version was introduced in Section X.1, where it was important to show that the appearance of the Newtonian potential in the non relativistic Hamiltonian originates in a weak metric field. This result implies that the theory has a high degree of uniqueness. The Lorentz invariant version of irrotational hydrodynamics is based on the action

$$A_1 = \int d^3x \left(\frac{\rho}{2}(g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - c^2) - W[\rho] \right), \tag{12.1}$$

with g the Lorentzian metric. The fully covariant form is exactly the same, with g the dynamical 4-dimensional, pseudo Riemannian metric. The two-step promotion of potential theory to general covariance is straight forward.

In Section XII.1, we shall do the same for the more general form of conservative hydrodynamics that was introduced in Chapter X. Actually, this will be done in reverse order, for the existing literature on relativistic field theories allows us to present the relativistic version in its fully developed form; then taking the non relativistic limit we gain a much more complete understanding of the vector field $\vec{X}$ and its dynamical role in hydrodynamics.

To relate non relativistic hydrodynamics to General Relativity we must pass through Special Relativity. All the successfull relativistic field theories are based on action principles; they are theories of elementary particles, more especially gauge theories. The very high degree of developments of this field explains the paradoxical fact that relativistic gauge theories are more familiar than non relativistic ones.

XII.1. The relativistic, antisymmetric tensor field

What is needed is a chapter from the theory of relativistic quantum gauge theories. Quantum? Yes, because a full understanding of gauge theories is reached only by quantization. That is too much for a book on thermodynamics, and for this reason what follows can only be a sketch. The reader is encouraged to read this short summary and follow up by a study of the literature, beginning with the paper by Ogievetskij and Polubarinov.

A dramatic effect of relativization is that the fields become propagating. In particular, propagation of the fields into empty space may be unavoidable, therefore it is prudent to keep the number of propagating modes at a minimum. And this too leads us to the antisymmetric tensor gauge field and its well known properties. This field has at most one scalar, propagating mode. It was introduced by Ogievetskij and Polubarinov (1964). In another context it is known as the B-field, or the Kalb-Ramond field. The connection to vorticity was foreseen by Lund and Regge(1976).

Let $Y = (Y_{\mu\nu})$ be an antisymmetric tensor field (a 2-form) and consider the Lagrangian density $\rho_0 dY^2$, with ρ_0 constant and [1]

$$dY^2 = \frac{c^2}{4} g^{\mu\mu'} g^{\nu\nu'} g^{\lambda\lambda'} Y_{\mu\nu,\lambda} \sum_{\text{cyclic}} Y_{\mu'\nu',\lambda'}. \tag{12.1.1}$$

[1] Units are Gaussian, slightly modified as explained later. All components of 4-tensors have the same dimension, for example $\partial_\mu = (\vec{x}, ct)$. The fields X, Y, η are lengths and the fields $\dot{\vec{X}}, \vec{E}, \vec{B}$ are velocities.

This involves the two-form field alone, there is no matter present, so far. Greek indices run over 1,2,3,0, latin indices over 1,2,3. The (inverse) metric tensor is Lorentzian, diagonal with $g^{11} = g^{22} = g^{33} = 1, g^{00} = -1/c^2$. It is invariant under the gauge transformation

$$\delta Y = d\xi.$$

In a 3-dimensional notation,

$$X^i = \frac{1}{2}\epsilon^{ijk}Y_{jk}, \quad \eta_i = Y_{0i},$$

The equation $\eta_i = \partial_0\xi_i - \partial_i\xi_0$ can always be solved for the vector field $\vec{\xi}$; there is a family of gauges in which the field $\vec{\eta}$ vanishes. In addition, the vector field $\vec{X}$ can be reduced to a gradient field. This spinless field is the only propagating mode.[2]

In flat space, with Cartesian coordinates, in terms of $\vec{X}$ and $\vec{\eta}$,

$$dY^2 = \frac{1}{2}\Big(\dot{\vec{X}} + c\vec{\bigtriangledown} \wedge \vec{\eta}\Big)^2 - \frac{c^2}{2}(\vec{\bigtriangledown} \cdot \vec{X})^2. \tag{12.1.2}$$

The free field equations associated with this expression for the Lagrangian density are

$$\frac{d}{dt}\Big(\dot{\vec{X}} + c\vec{\bigtriangledown} \wedge \vec{\eta}\Big) - c^2\vec{\bigtriangledown}\,(\vec{\bigtriangledown} \cdot \vec{X}) = 0, \quad \vec{\bigtriangledown} \wedge (\dot{\vec{X}} + c\vec{\bigtriangledown} \wedge \vec{\eta}) = 0. \tag{12.1.3}$$

The only mode that propagates *in vacuo* is a spinless mode, but we can add sources,

$$\frac{d}{dt}\Big(\dot{\vec{X}} + \vec{\bigtriangledown} \wedge \vec{\eta}\Big) - c^2\vec{\bigtriangledown}(\vec{\bigtriangledown} \cdot \vec{X}) = \vec{K}, \quad \vec{\bigtriangledown} \wedge (\dot{\vec{X}} + c\vec{\bigtriangledown} \wedge \vec{\eta}) = \vec{K}', \tag{12.1.4}$$

so that, in their presence, the field $\dot{\vec{X}}$ need not be be irrotational.

The term $\rho\dot{\vec{X}} \cdot \vec{\bigtriangledown}\Phi$ in our non relativistic Lagrangian can be made invariant under Lorentz transformtions by expanding it to

$$\rho\frac{c}{2}\epsilon^{\mu\nu\lambda\rho}Y_{\mu\nu,\lambda}\psi_{,\rho}. \tag{12.1.5}$$

[2] For dimensional reasons a type of density factor is required in any field theory Lagrangian. In hydrodynamics the mass density serves naturally, in electromagnetism the role is taken by the susceptibility of the vacuum. the densities $\rho, \epsilon, \gamma t$ all have the same dimension. We shall take up this question when it becomes important to do so.

Here ψ is a Lorentz scalar. With $\rho = \rho_0$, uniform, this term is "topological'; that is, it is a boundary term and can be ignored.

Introduce the dual, $\tilde{Y}^{\mu\nu} = \epsilon^{\mu\nu\sigma\tau} Y_{\sigma\tau}$, then

$$\frac{c}{2}\epsilon^{\mu\nu\lambda\rho} Y_{\mu\nu,\lambda}\psi_{,\rho} = c^2 \tilde{Y}^{\lambda\rho}_{,\lambda}\psi_{,\rho} = c(\tilde{Y}_{0i,0}\psi_{,i} + \tilde{Y}_{i0,i}\psi_{,0} - \tilde{Y}_{ij,i}\psi_{,j}),$$

$$\tilde{Y}_{i0,i} = \vec{\bigtriangledown} \cdot \vec{X}, \quad \tilde{Y}_{ij,i} = (\vec{\bigtriangledown} \wedge \vec{\eta})_j,$$

and

$$\frac{c}{2}\epsilon^{\mu\nu\lambda\rho} Y_{\mu\nu,\lambda}\psi_{,\rho} = \dot{\vec{X}} \cdot \vec{\bigtriangledown}\psi - (\vec{\bigtriangledown} \cdot \vec{X})\dot{\psi} + c(\vec{\bigtriangledown} \wedge \vec{\eta}) \cdot \vec{\bigtriangledown}\psi. \tag{12.1.6}$$

The first term on the right is the one that appears in the non relativistic theory. In the second term $\dot{\psi}$ appears in the role of a Lagrange multiplier and imposes the additional constraint

$$\vec{\bigtriangledown} \cdot \vec{X} = 0.$$

The Lagrangian density investigated by Ogievetskij and Palubarinov include electromagnetic fields and a mixing term,

$$\rho_0 dY^2 + \epsilon_0 F^2 + \gamma Y F. \tag{12.1.7}$$

The last term is a 4-form, an outer product of two 2-forms,

$$\gamma Y F := \gamma(\vec{B} \cdot \vec{\eta} + \vec{E} \cdot \vec{X}), \tag{12.1.8}$$

and γ is uniform. This is not an interaction, but a mixing term; the effect is to give mass to the photon. the easiest way to see this is to use the transverse gauge, in which

$$\partial_\mu Y_{\mu\nu} = \partial_\mu A_\mu = 0.$$

In that case the field equations are

$$\delta\vec{X} \cdot \left(\frac{d}{dt}\rho_0((\dot{\vec{X}} + c\vec{\bigtriangledown} \wedge \vec{\eta})) - \gamma\vec{E}\right)$$

$$+\delta\vec{\eta} \cdot \left(\vec{\bigtriangledown} \wedge (\rho_0(\dot{\vec{X}} + c\vec{\bigtriangledown} \wedge \vec{\eta}) - \frac{\gamma}{c}\vec{B}\right)$$

$$+\delta\vec{E}\Big(\epsilon_0\vec{E}+\gamma\vec{X}\Big)+\delta\vec{B}\cdot\Big(\epsilon_0\vec{B}+\gamma\vec{\eta}\Big)+\delta AJ=0.$$

Here $\vec{E}=\vec{\bigtriangledown}A_0-\dot{\vec{A}}/c,\quad \vec{B}=c\vec{\bigtriangledown}\wedge\vec{A}$ and the last line transforms by an integration by parts to

$$-\delta A_0\Big(\epsilon_0\vec{\bigtriangledown}\cdot\vec{E}+\gamma\vec{\bigtriangledown}\cdot\vec{X}-J_0\Big)+\delta\vec{A}\cdot\Big(\epsilon_0\dot{\vec{E}}+\gamma\dot{\vec{X}}-\vec{J}\Big)-c\delta\vec{A}\cdot\vec{\bigtriangledown}\wedge\Big(\epsilon_0\vec{B}+\gamma\vec{\eta}\Big).$$

Photon mass

A formal integration over the tensor field Y shows that this theory is dual to massive electrodynamics. We wish to reach this conclusion by more elementary means. To this end we must eliminate the tensor field from the field equations, the goal being to relate the electromagnetic potential directly to the sources. This requires that we fix the gauge. The gauge of the tensor theory has already been fixed by setting $\eta_i=Y_{i0}=0, i=1,2,3.$

The relevant part of the Lagrangian density is

$$\frac{\epsilon_0}{2}(\vec{E}^2-\vec{B}^2)+\gamma(\vec{E}\cdot\vec{X}+\vec{B}\cdot\cdot\vec{\eta}).$$

Here $\vec{E}=\dot{\vec{A}}-c\vec{\bigtriangledown}A_0,\ \vec{B}=c\vec{\bigtriangledown}\wedge\vec{A}$ and the variation is

$$(\epsilon_0\vec{E}+\gamma\vec{X})\cdot(\delta\dot{\vec{A}}-c\vec{\bigtriangledown}\delta A_0)+(-\epsilon_0\vec{B}+\gamma\vec{\eta})\cdot(c\vec{\bigtriangledown}\wedge\delta\vec{A})$$

$$=(-\epsilon_0\dot{\vec{E}}-\gamma\dot{\vec{X}}+c\epsilon_0\vec{\bigtriangledown}\wedge\vec{B})\cdot\delta\vec{A}+c(\epsilon_0\vec{\bigtriangledown}\cdot\vec{E}+\gamma\vec{\bigtriangledown}\cdot\vec{X})\delta A_0.$$

It follows that

$$\vec{\bigtriangledown}\cdot\vec{E}+\gamma\vec{\bigtriangledown}\cdot\vec{X}=0 \tag{12.1.9}$$

and that

$$\epsilon_0(\,-\ddot{\vec{A}}+c\vec{\bigtriangledown}\dot{A}_0+c^2\vec{\bigtriangledown}(\vec{\bigtriangledown}\cdot\vec{A})-c^2\Delta\vec{A})-\gamma\dot{\vec{X}}=0. \tag{12.1.10}$$

This last equation can be written

$$\epsilon_0(-\Box\vec{A}+c^2\vec{\bigtriangledown}\Lambda)=\gamma\dot{\vec{X}}, \tag{12.1.11}$$

the 4-divergence $\Lambda=\partial\cdot A$ can be chosen arbitrarily, thus fixing the electromagnetic gauge. We must use the field equations, from variation of $\vec{X}$ and $\vec{\eta}$,

$$\rho_0\ddot{\vec{X}}=\gamma\vec{E},\qquad \rho_0\vec{\bigtriangledown}\wedge\dot{\vec{X}}=\gamma\vec{B},$$

to eliminate the field $\vec{X}$ from Eq.s (12.1.9-10). These equations tell us that, in empty space, $\dot{\vec{X}} = (\gamma/\rho_0)\vec{A}$, *modulo* a gradient. Finally, there is a gauge in which (12.1.11) reduces to

$$-\Box\vec{A} = \frac{\gamma^2}{\epsilon_0\rho_0}\dot{\vec{A}}.$$

The photon is thus massive, with 'mass'

$$m_{ph}c^2 = \frac{\gamma\hbar}{\sqrt{\epsilon_0\rho_0}}.$$

Planck's constant has the value $\hbar = 1.05510^{-34}J.s = 6.5810^{-16}eV.s$.

In this formula the "constants" are: Only γ is a true constant; it must be so by gauge invariance. The electromagnetic density ϵ is a constant in vacuum but it will have a different value in a homogeneous environment with matter in it. Finally ρ_0 is the local density of matter. All are taken to be uniform in the calculation of the photon mass.

The covariant gauge

The equations are much easier to deal with in the covariant gauge,

$$\partial^\mu Y_{\mu\nu} = 0, \quad \partial^\mu A_\mu = 0.$$

In empty space, when ρ is uniform, the equations of motion take the form

$$\rho_0 p^2 Y_{\mu\nu} + \gamma\epsilon_0\tilde{F}_{\mu\nu} = 0, \quad \epsilon p^2 A_\nu + \rho_0\gamma p^\mu\tilde{Y}_{\mu\nu} = 0.$$

If the momentum vector is timelike we can go to the rest system, where $p = (m, 0, 0, 0)$ and the two equations take the form

$$\rho_0 m\tilde{Y}_{ij} + \gamma_0\epsilon A_{ij} = 0, \quad \epsilon m A_{ij} + \rho_0\gamma\tilde{Y}_{ij} = 0.$$

Here we can solve the first equationfor the Y field and substitute in the second to get

$$Y_{ij} = \frac{-\gamma}{m^2}\tilde{F}_{ij},$$

and eliminate it to get

$$m^2 = \frac{\hbar^2\gamma^2}{\epsilon_0\rho_0}.$$

The effect of a photon mass is spectacular, unless it is extremely small. Many studies have been made to establish an upper limit, one as low as $10^{-38}ev/c$. But in the presence of important, rotating masses the estimate may have to be revised.

Conservative Hydrodynamics

Now we consider the total Lagrangian density

$$\mathcal{L} = \frac{\rho}{2}(g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}-c^2)+\rho dY^2+\kappa\rho\frac{c^2}{2}\epsilon^{\mu\nu\lambda\rho}Y_{\mu\nu,\lambda}\psi_{,\rho}+\gamma YF-f-sT \quad (12.1.12)$$

and the action $\mathcal{A} = \int d^3xdt\,\mathcal{L}$. The first term is the Lorentz invariant contribution that appears in the non relativistic approximation as the Galilei covariant field $\rho(\dot\Phi - \vec\bigtriangledown\Phi^2/2 - \varphi)$.

The variation of the action with respect to ψ is $-\delta\psi$ times

$$\frac{d}{dt}\Big(\rho(\frac{\dot\psi}{c^2}-\kappa\vec\bigtriangledown\cdot\vec X)\Big)+\vec\bigtriangledown\cdot\Big(\rho(\kappa\dot{\vec X}-\vec\bigtriangledown\Phi+\kappa\vec\bigtriangledown\wedge\vec\eta)\Big) =: \frac{d}{dt}J^0+\vec\bigtriangledown\cdot\vec J. \quad (12.1.13)$$

In the non relativistic limit $\dot\psi/c^2$ is unity. The boundary conditions require that the current $\vec J$ be normal to the boundary. the last equation reduces to the equation of continuity. In a physical gauge

$$\dot\rho + \vec\bigtriangledown\cdot\Big(\rho(\kappa\dot{\vec X} - \vec\bigtriangledown\Phi)\Big) = 0.$$

This confirms that the conserved flow is

$$\rho\vec v = \rho(\kappa\dot{\vec X} - \vec\bigtriangledown\Phi).$$

The variation of the action with respect to the field $\vec X$ is

$$\int d^3xdt\,\delta\vec X\cdot\Big(\frac{d}{dt}(\rho(\dot{\vec X}+\kappa\vec\bigtriangledown\psi+\vec\bigtriangledown\wedge\vec\eta))+\vec\bigtriangledown(\rho(\kappa-c^2\vec\bigtriangledown\cdot\vec X))-\gamma\vec E\Big). \quad (12.1.16)$$

Setting this to zero gives the field equation

$$\frac{d}{dt}(\rho(\dot{\vec X} + \kappa\vec\bigtriangledown\Phi + \vec\bigtriangledown\wedge\vec\eta)) + \vec\bigtriangledown(\rho(\kappa - c^2\vec\bigtriangledown\cdot\vec X)) = \gamma\vec E. \quad (12.1.17)$$

An electric field is expected on the basis of an experiment by Tolman (1910). It may also have something to do with the anomalous Seebeck effect. But at this time we are far from understanding all the ramifications of this interaction with the electromagnetic field. That an unexpected role may be played by the electric field in connection with vortex motion, or that an electromagnetic analogy may be glimpsed here, was suggested by Feynman in connection with liquid helium (Feynman 1954 page 273).

The variation of the action with respect to the field $\vec{\eta}$ is

$$\int d^3x dt\, \delta\vec{\eta} \cdot \Big(\vec{\bigtriangledown} \wedge (\rho(\dot{\vec{X}} + \kappa\vec{\bigtriangledown}\Phi + \vec{\bigtriangledown} \wedge \vec{\eta}) - \frac{\gamma}{c}\vec{B}\Big). \tag{12.1.16}$$

Setting this to zero gives the onstraintn

$$\vec{\bigtriangledown} \wedge (\rho(\dot{\vec{X}} + \kappa\vec{\bigtriangledown}\Phi + \vec{\bigtriangledown} \wedge \vec{\eta}) = \frac{\gamma}{c}\vec{B} \tag{12.1.17}$$

In the non relativistic context the gauge is fixed and the field $\vec{\eta}$ is absent; but the constraint must be taken into account. It is this constraint, somewhat mysterious in the non-relativistic context, that reduces the number of degrees of freedom from 3 to 1.

The two equations (12.2.11) and (12.2.13 are mutually consistent by Maxwell's second equation, $dF = 0$ or $\dot{\vec{B}} = c\vec{\bigtriangledown} \wedge \vec{E}$, provided that γ is constant (as is required by the gauge invariance of (12.1.7). The new term is nevertheless ignorable, since it can be absorbed into $\dot{\Phi}$.

In the Lorentz invariant gauge theory the metric is fixed, Lorentzian; the action becomes generally coordinate invariant when the fixed metric is replaced by a dynamical, pseudo Riemannian metric field.

XII.2. Non relativistic limit and Galilei transformations

A concept of a non-relativistic limit of a relativistic field theory can be envisaged if each of the dynamical variables can be represented as a power series in $1/c$, beginning with a term of order zero; that is, $(1/c)^0$, or higher, $(1/c)^1, (1/c)^2, ...$'. In the case of the model considered we must assume that this is the case for the variables ρ, Y, ψ. The non-relativistic limit of the Lagrangian exists if every term is of positive or zero order. Dropping all terms of positive order we may ask about the physical meaning of the remainder, including transformation properties.

Taking the basic variables to be ρ, Y and ψ we find that there is one term that is visibly of order c^2, in the kinetic term (12.1.2). To overcome this obstruction we must postulate a boundary condition

$$\vec{\bigtriangledown} \cdot \vec{X} = c^{-2}\Theta + o(c^{-3}), \tag{12.2.1}$$

with Θ of order 0 in $1/c$.

The κ term introduces another term of nominal order c^2 as is seen in Eq.(12.1.7) since $\dot{\psi} = c^2 + \dot{\Phi}$,

$$-(\vec{\bigtriangledown} \cdot \vec{X})\dot{\psi} = -\Theta + o(1/c).$$

We conclude that the existence of a non-relativistic limit of the Lagrangian depends on the validity of (12.2.1).

We can now ask about the Galilei invariance of the bi-vector theory. The subgroup of 'proper' Galilei transformations derives from Lorentz transformations of the form

$$\delta\vec{x} = t\vec{u}\gamma, \quad \delta t = (\vec{u} \cdot \vec{x}/c^2)\gamma, \quad \gamma = \frac{1}{\sqrt{1-(u/c)^2}}.$$

Infinitesimal Galilei transformations are related to first order Lorentz transformations. It is enough to retain terms linear in $\vec{u}$, replacing γ by unity. Infinitesimal Galilei transformations are defined as the 'contraction' that consists of taking the limit $c \to \infty$.But it would be imprudent to take that limit already at this stage, as we shall see. So the transformations to be considered are first order or infinitesimal Lorentz transformations,

$$\delta\vec{x} = t\vec{u}, \quad \delta t = \vec{u} \cdot \vec{x}/c^2.$$

In what we shall call a physical gauge the field $\vec{\eta}$ vanishes. The Lorentz group acts on the antisymmetric field in the manner that is indicated by the indices, in particular

$$\delta Y_{ij} = t\vec{u} \cdot \vec{\bigtriangledown} Y_{ij} + tu_i Y_{0j} + tu_j Y_{i0}, \quad \delta Y_{0j} = t\vec{u} \cdot \vec{\bigtriangledown} Y_{0j} + u_i Y_{ij}/c^2.$$

Except for the first, the terms that are linear in t vanish in a physical gauge, so each component Y_{ij} and each component of $\vec{X}$ transforms as a three-dimensional scalar field under Galilei transformations. The last term is another matter, its presence shows that the transformed field is not in the physical gauge, since $\delta Y_{0j} \neq 0$. So we have to make a gauge transformation

$$\delta Y_{ij} = \partial_i \xi_j - \partial_j \xi_j, \quad \delta Y_{0j} = \partial_0 \xi_j - \partial_j \xi_0,$$

such that

$$\partial_0 \xi_j - \partial_j \xi_0 = -u_i Y_{ij}/c^2.$$

The very existence of a non relativistic limit implies that the basic variables can be represented as power series in $1/c$. So even if this transformation implies a change in Y_{ij}, that will be a change of order $1/c^2$ and we can be fairly confident that this is enough that this change can be ignored. We shall see below that great caution is necessary and it is worth while to point out that if $\vec\bigtriangledown\cdot\vec X$ is of order $1/c^2$ then the change δY_{ij} will be of order $1/c^4$. To see this take the curl of the last equation to get

$$\partial_0\vec\bigtriangledown\wedge\vec\xi = \vec u(\vec\bigtriangledown\cdot\vec X)/c^2.$$

If, as we suppose, $\vec\bigtriangledown\cdot\vec X$ is of order $1/c^2$ then this makes ξ of order $1/c^4$. We conclude that Galilei transformations affect the field $\vec X$ only the way that it affects scalar fields, by the argument shift $\delta\vec x = \vec u t$.

The reason why we are proceeding cautiously is that there are terms in the Lagrangian of apparent order c^2, see (12.1.2),

$$\rho\Big(-\frac{c^2}{2}(\vec\bigtriangledown\cdot\vec X)^2\Big) \tag{12.2.2}$$

and (12.1.7)

$$\rho\kappa\Big(-(\vec\bigtriangledown\cdot\vec X)\dot\psi\Big),\quad \dot\psi = c^2+\dot\Phi. \tag{12.2.3}$$

Therefore, in order for the non-relativistic limit to exist it is required that $\vec\bigtriangledown\cdot\vec X$ be of order at least 2 in $1/c$.

The first of the two dangerous terms is ignorable and the other reduces Mod higher order terms to

$$-\rho\kappa\Theta.$$

To sum up, for the existence of the non-relativistic limit it is necessary and sufficient that $\vec\bigtriangledown\cdot\vec X$ is of order ≥ 2 in $1/c$, and the only correction needed in the non-relativistic Lagrangian discussed earlier is the addition of $-\rho\kappa\Theta$, the field Θ defined as

$$\Theta := \lim_{c\to\infty} c^2\vec\bigtriangledown\cdot\vec X.$$

This restores the incariance under Galilei transformations that was lost with the introoduction of the κ term in the Lagrangian.

Unfortunately this quantity Θ cannot be calculated within the non-relativistic theory. The relativistic equation of continuity, obtain by variation of ψ, yields

$$\dot\rho\vec\bigtriangledown\cdot\vec X = \dot\rho + \vec\bigtriangledown\cdot(\rho(\kappa\dot{\vec X} - \vec\bigtriangledown\Phi) = 0.$$

The non-relativistic conservation law must be valid up to corrections of order $1/c^2$. These corrections are encoded in the value of the field Θ and this introduces a degree of arbitrariness into the theory. The field Θ is an independent field although its variation under Galilei transformations is related to that of $\vec{X}$, making

$$\Theta + \dot{\vec{X}} \cdot \vec{\bigtriangledown}\Phi$$

invariant. Invariance under Galilei transformations is thus formal; the choice $\Theta = 0$ characterizes a preferred Galilei frame of reference. This is an unsatisfactory result of including the κ term. We shall retain it nevertheless, for it has turned out to be essential. More work is required to settle questions that remain.

The total, non-relativistic Lagrangian density is thus

$$\rho\Big(\dot{\Phi} - \Theta(1+\kappa) + \dot{\vec{X}}^2/2 + \kappa\dot{\vec{X}} \cdot \vec{\bigtriangledown}\Phi - (\vec{\bigtriangledown}\Phi)^2/2\Big) - f - sT.$$

The field θ plays the same role as the B field in electromagnetism; the theory, in the non relativistic approximation, is not manifestly Galilei invariant.

XII.3. Dark Matter

in order that the new Conservative H drodynamis be generally useful it is important that the vorticity field $\vec{X}$ have a universal coupling to matter. If that includes electromagnetism then one effect will be to regularize the infrared. The Higgs boson will no longer be needed.

Another immediate effect will be another source of matter to the population of the universe; the new field will clearly be contributor to Dark Matter. It would seem reasonable to expect that the greatest field strengths will be found in the neighborhood of clusters of galaxies and that they may be sufficiently strong to be recorded in experiments on gravitational lensing. A still more promising direction is the study of the Milky Way (and other galaxies) and the curious, non-Newtonian stellar orbits that are being observed.

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