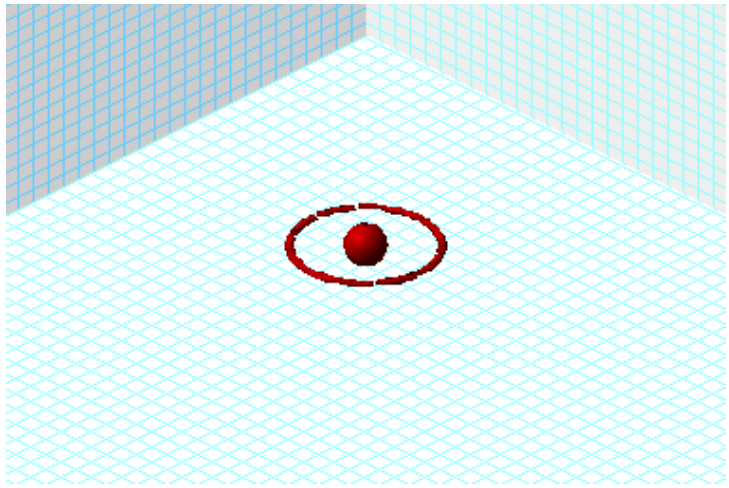


XI. Rotating planets.

There is a limit to how far one can develop astrophysics as an application of non relativistic hydrodynamics; here is an attempt that pushes that limit.

Variational techniques have been used in applications of hydrodynamics in special cases but an action that is general enough to deal with both potential flows and solid-body flows, such as cylindrical Couette flow and rotating planets, has been proposed only recently. This paper is one of a series that aims to test and develop the new Action Principle. We study a model of rotating planets, a compressible fluid in a stationary state of motion, under the influence of a fixed or mutual gravitational field. The main problem is to account for the shape and the velocity fields, given the size of the equatorial bulges, the angular velocity at equator and the density profiles. The theory is applied to the principal objects in the solar system from Earth and Mars to Saturn with fine details of its hexagonal flow and to Haumea with its odd shape. With only 2 parameters the model gives a fair fit to the shapes and the angular velocity field near the surface. Planetary rings are an unforeseen, but a natural and inevitable feature of the dynamics; no cataclysmic event need be invoked to justify them. The simple solutions that have been studied so far are most suitable for the hard planets, and for them the predicted density profiles are reasonable. The effect of precession was not taken into account, nor were entropic forces, so far. There has not yet been a systematic search for truly realistic solutions. The intention is to test the versatility of the action principle; the indications are are very encouraging.

XI.1. Introduction

The ultimate aim of our work is to learn how to deal with a compressible, rotating fluid in General Relativity. Because much important work has been done in the past, we must emphasize the novelty of this work: We approach hydrodynamical problems within the framework of a general **Action Principle**. In the context of General Relativity this means selecting an adequate relativistic action A_{matter} for fluid dynamics in an arbitrary metric background and adding it to Einstein's action for the metric. The energy momentum tensor of A_{matter} becomes the source of Einstein's field equations. This is the natural way to ensure that the source of Einstein's field equations satisfy the integrability conditions (the Bianchi constraint).

Our action principle incorporates traditional hydrodynamics, **including the equation of continuity**. It generalizes a well known action principle for irrotational hydrodynamics by including rotational velocity fields. It extends to Thermodynamics where it has been extensively applied to mixtures (Fronsdal 2018), to Special Relativity (Ogievetskij and Polubarinov 1964), and most recently to stability of cylindrical Couette flow (Fronsdal 2018). It provides the first field theoretic model of rotating fluids that respects the Bianchi identity and includes the equation of continuity. Previous attempts to formulate an action principle for General Relativity (Hartle 1967, Bardeen 1970, Schutz 1970 and Taub 1954) have not been developed to a stage where applications could be undertaken, but all the ideas of the older work resonate in the present one.

The principal feature of this paper is the action principle; in other respects it is far less complete than earlier models of planetary dynamics. See for example Beauvalet, Lainey, Arlot and Binzel (2012) or Stute, Kley and Mignone (2013). Another feature that is not always included in planetary dynamics is the requirement that the mass flow velocity field be harmonic, as it needs to be for all stationary flows. (See below.) Most important, the action principle is a framework with a greatly improved predictive power, as this paper demonstrates.

Before attacking this problem in the full, Generally Relativistic setting, it is good strategy to study the problem in Newtonian Gravity. The system is a **compressible fluid** in hydrodynamics; the forces arise from a fixed, central Newtonian potential, or from the mutual, gravitational interaction between the fluid elements and the pressure. Because the new Action Principle is a

new departure, we present a summary account of it, before applying it to the principal components of the solar system. Harmonic expansions of the dynamical fields will include only the first three terms, except in a study of the hexagonal flows in Saturn.

The gauged-fixed, non-relativistic action is

$$A = \int dt d^3x \mathcal{L}, \quad \mathcal{L} = \rho(\dot{\Phi} - \vec{\nabla}\Phi^2/2 + \dot{\vec{X}}^2/2 + \kappa\dot{\vec{X}}\cdot\Phi - \Phi^2/2 - \varphi) - W[\rho]. \quad (1.1)$$

The variables are the density ρ and two velocity potentials, the scalar velocity potential Φ and the vector potential $\vec{X}$. (Compare Schutz 1970.) Projected on the theory of rotational flows, the special case $\dot{\vec{X}} = 0$, it becomes the Lagrangian that was discovered by Lagrange himself (1760) and brought to our attention by Lamb (1932) and by Fetter and Walecka (1980). Gravity is represented by the Newtonian potential φ; it can be a fixed, central potential or the mutual energy of interaction between fluid elements. The action (1.1) is the non relativistic limit of a generally relativistic action and the potential $W[\rho]$ is the thermodynamic internal energy density for a system with fixed, uniform specific entropy density. The on-shell value of the Lagrangian density is the thermodynamic pressure; as prophesized with rare insight by Taub (1954).

The Euler-Lagrange equations are: the equation of continuity (from variation of the velocity potential Φ),

$$\dot{\rho} + \vec{\nabla} \cdot (\rho\vec{v}) = 0, \quad \vec{v} := \kappa\dot{\vec{X}}X - DD\Phi, \quad (1.2)$$

the wave equation (from variation of $\vec{X}$)

$$\frac{d}{dt}(\rho\vec{w}) = \vec{\nabla} \wedge \vec{v}, \quad \vec{w} := \dot{\vec{X}} + \kappa\vec{\nabla}\Phi, \quad (1.3)$$

and the Bernoulli equation (from variation of ρ),

$$\dot{\Phi} + \dot{\vec{X}}^2/2 + \kappa\dot{\vec{X}} \cdot \Phi - \Phi^2/2 - \varphi = \frac{\partial}{\partial\rho}W[\rho]. \quad (1.4)$$

The incorporation of two velocity fields is an essential feature of the theory; what is both novel and effective is that they contain the correct (minimal) number of dynamical variables: 4 for hydrodynamics including

the density. The Lagrangian has one free parameter κ; it is related the compressibility of the fluid.

The complete theory is a relativistic gauge theory; here we are working in the physical gauge and in the non relativistic limit. There are two velocity fields, $\vec{v}$ represents mass flow; $\vec{w}$ allows for vorticity. A key feature of the theory is the constraint

$$\vec{\nabla} \wedge (\rho \vec{w}) = 0, \tag{1.5}$$

derived by variation of the full Lagrangian with respect to a vector gauge field. (The gauge field is fixed at zero and does not appear in this paper.) It implies that the vector field $\vec{w}$ - defined in (1.3) - can be expressed in terms of a scalar field,

$$\vec{w} = \frac{-1}{\rho} \vec{\nabla} \tau. \tag{1.6}$$

(Of course, we cannot replace $\vec{w}$ by τ in the action.) The limit of an incompressible fluid is approached as $\kappa \to \infty$. It is this constraint that connects the velocity and the density: in the context of the action principle harmonicity of $\vec{v}$ becomes a constraint on ρ, solved by Eq.(2.8) below.

We have given a brief account of the theory in the physical gauge; the full gauge theory is described in two papers (Fronsdal 2014 and 2017).

What does the most to validate this theory is that it is formulated as an action principle. To account for rotational motion it includes the velocity field $\dot{\vec{X}}$, as in 'Lagrangian hydrodynamics', but as this would seem to add 3 additional degrees of freedom, we need a constraint that effectively reduces this number to 1, as is accomplished by Eq.(1.5). The relation of the field $\dot{\vec{X}}$ to vorticity, and to a relativistic gauge theory, was proposed, in the special case of incompressible flows, by Lund and Regge (1976). In string theory it is the Kalb-Ramond field (Kalb and Ramond 1974).

The action is not completely new; hints of it appeared in a classical paper by Hall and Vinen (1956) on superfluids and in a more recent review by Fetter (2009) on rotating Bose-Einstein condensates (Fetter (2009). [1] In those papers $\dot{\vec{X}}$ is not a local, dynamical field variable but a fixed backgound feature that accounts for a rigid rotation of the whole system. Although the dynamical, irrotational velocity was seen as insufficient and another degree

[1] The equations of motion in the paper by Hall and Vinen (1956) are widely quoted; the relation of their work to an action principle not at all.

of freedom was needed, the way to avoid an excessive number of degrees of freedom, by means of the constraint (1.5), was not widely known. The field components $\dot{X}^i$ appear in Bardeen (1970), they are the classical 'Lagrange parameters'.

A stationary flow is one that evades the dissipating effect of **viscosity**. In traditional hydrodynamics viscosity is included as an additional term in the Navier-Stokes equation,

$$\dot{\vec{v}} + (\vec{v} \cdot \vec{\bigtriangledown})\vec{v} = \frac{-1}{\rho}\vec{\bigtriangledown}p + \bar{\mu}\rho\Delta\vec{v}, \tag{1.7}$$

where $\bar{\mu}$ is the kinematical viscosity, usually a constant. Viscosity cannot be accommodated within an action principle, but its effect can be acknowledged by replacing the conservation law (1.3) by

$$\frac{d}{dt}(\rho\vec{w}) = \bar{\mu}\rho\Delta\vec{v}.$$

In this way a theory based on conservation laws is distorted to include a type of dissipation, precisely as is done in the familiar approach with Eq.(1.7). In both theories, **stationary motion is possible only when the field $\vec{v}$ is harmonic**.

The biggest surprise to emerge from this work is the natural and spontaneous appearance of planetary rings. Rings are predicted, not just accommodated. The simplest solutions studied in this paper predict a single ring (or none) for each planet. It is suggested that the parameter N in Eq.(1.9) is related to evolution, that all the planets may have had rings at one time, that the planet Mars, in particular, may have had a ring at relatively recent times and that all the planets will eventually loose their rings. The observed hexagonal rotation pattern of Saturn is useful data that reveals the presence of higher harmonics. The observation of 1-6 internal whorls is a prediction of the theory.

A brief digression on the cylindrical Couette problem

The most familiar systems treated in Newtonian gravity as well as in General Relativity have spherical symmetry. But most heavenly bodies are rotating around an axis that is more or less fixed, with approximate cylindrical symmetry. Much of the inspiration for our work comes from a study of cylindrical Couette flow. Following the advice of Homer Lane (1870), as is traditional in astrophysics, we apply to astrophysics what we have learned

in terrestrial laboratories. The problem examined by Couette (1888-1890) and Mallock (1888, 1896) is a fluid confined between two concentric cylinders that can be rotated independently around the vertical axis. In classical hydrodynamics the balance of forces is expressed by the Navier-Stokes equation. Boundary conditions, validated by the analysis of Taylor (1923), is assumed to be non-slip and the fluid is compressible. At low speeds any stationary motion is found to be described by the following harmonic vector field

$$\vec{v} = \frac{a}{r^2}(-y, x, 0) + b(-y, x, 0), \quad r := \sqrt{x^2 + y^2}, \quad a, b \text{ constant.} \tag{1.8}$$

The first term is irrotational for $r \neq 0$ and both are harmonic.

The new action principle was used to account for the stability of basic, cylindrical Couette flow (Fronsdal 2018). One feature of the model is that the density profile is subjected to a strong condition that originates in the demand that $\vec{v}$ be harmonic.

XI.2. A compressible fluid rotating in a fixed, central gravitational field

Our model of a planet is an isolated system with a liquid or solid core in thermodynamic equilibrium with a gaseous atmosphere, in a stationary, rotating state and described by the action (1.1). In the simplest case it consists of a single substance in two phases. The condition of thermodynamic equilibrium at the phase boundary is that the pressure p, the temperature and the chemical potential μ be continuous across the surface. In the case of a thin atmosphere this implies that the pressure and the chemical potential are constant on the surface. The surface is thus a locus of the function - see Eq.(1.4) -

$$C(\vec{x}) := \dot{\vec{X}}^2/2 + \kappa \dot{\vec{X}} \cdot \vec{\bigtriangledown}\Phi - \vec{\bigtriangledown}\Phi^2/2 - \frac{-GM}{R}, \quad R := \frac{r}{|\sin\theta|} = \sqrt{x^2 + y^2 + z^2}. \tag{2.1}$$

We have chosen the Newtonian expression for the attractive gravitational potential. It is an expedient shortcut of the present treatment, as in the simplest version of the traditional approach, and one that we hope to remove later. It is a valid approximation so long as the departure from spherical symmetry is small.

This model should be appropriate for Earth and Mars and possibly for the frozen planets Neptune and Uranus, less so for the gaseous planets. To determine the appropriate velocity fields we begin by examining the simplest solutions.

If the velocity is irrotational, and $\dot{\vec{X}} = 0$; the shape is determined by

$$-\frac{a^2}{2}\frac{1}{r^2} + \frac{GM}{R} = \text{constant}, \qquad r := R|\sin\theta| = \sqrt{x^2+y^2}.$$

We plot the loci of this expression and vary the parameter a^2. Instead of an equatorial bulge there is a polar hole, see Fig.3.1. This attempt evidently fails.

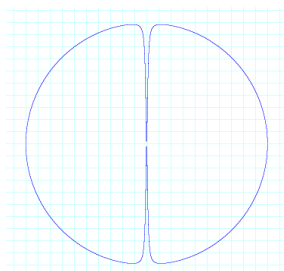


Fig.2.1b. Showing the 'equatorial bulge', or rather the polar depression, that would be the effect of pure, irrotational flow.

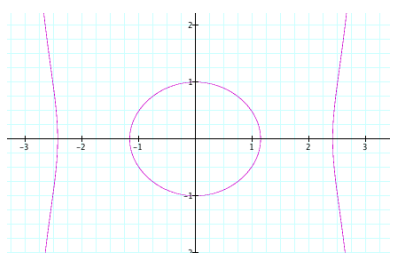


Fig.2.2. The more physical locus predicted by solid body flow. (The non compact branch is not part of the traditional theory.)

Solid-body flow is the complementary case in which $\vec{\nabla}\Phi = 0$ and the angular velocity $\omega = b$. As in the traditional approach; the condition of equilibrium is

$$\frac{\omega^2}{2}r^2 + \frac{GM}{R} = \text{ constant.}$$

There is a bulge, see Fig.2.2. The number usually quoted is

$$\epsilon := \frac{R_{eq.}}{R_{pole}} - 1 = \frac{\omega^2}{2MG} R_{eq.}^3,$$

where, to a good approximation, R_{eq} can be replaced by 1 on the right hand side. For Planet Earth the number is $MG = R^2 g,\ g = 998 cm/sec^2$, and the approximate value of ϵ is predicted by this model to be

$$\frac{\omega^2 R^3}{2MG} = \frac{R\omega^2}{2g} = \left(\frac{2\pi}{24 \times 3600}\right)^2 \frac{6.357 \times 10^8}{2 \times 998} = .0016. \tag{2.2}$$

Solving Eq.(2.1) for the azimuthal angle we find that $(b^2/2)R^2 \sin^2\theta = C - GM/R$ and we conclude that, with the solid body hypothesis the radius is minimal at the poles and the locus of (3.1) always has a non-compact branch, as in Fig.2.2.

The observed value of ϵ for Earth is .00335, more than twice the prediction (2.2). The classical theory can be improved by taken into account the effect of the bulge on the potential; for example, by assuming that the shape is an ellipsoid, and that the density is uniform. That results in a value for ϵ of .0042, which is too large (Fitzpatrick 2018). A further improvement results from taking the partly known density distribution into account; this has the effect of diminishing the effect of the shape on the potential.

The general case

We look for the general solution of the equations of motion. Again

$$\varphi = -\frac{GM}{R}, \quad \mathcal{R} := \sqrt{x^2 + y^2 + z^2}, \quad G = \text{ constant}. \tag{2.3}$$

The full set of equations includes the equation of continuity and

$$\vec{v} := \kappa \dot{\vec{X}} - \vec{\nabla}\dot{\vec{\Phi}}, \quad \Delta\vec{v} = 0, \quad \vec{w} := \dot{\vec{X}} + \kappa\vec{\nabla}\dot{\vec{\Phi}} = \frac{-1}{\rho}\vec{\nabla}\tau, \tag{2.4}$$

$$\dot{\Phi} + \dot{\vec{X}}^2/2 + \kappa\dot{\vec{X}} \cdot \vec{\nabla}\dot{\vec{\Phi}} - \vec{\nabla}\dot{\vec{\Phi}}^2/2 + \frac{GM}{R} = \mu[\rho], \tag{2.5}$$

All the vector fields can be expressed in terms of the two scalar fields Φ and τ and the density. The most naive assumption is that the two scalar fields are the same as those that appear in Couette flow, proportional to the azimuthal angle, thus

$$-\vec{\nabla}\Phi = \frac{a}{r^2}(-y, x, 0), \quad -\vec{\nabla}\tau = \frac{b}{r^2}(-y, x, 0). \tag{2.6}$$

These are gradient-type vector fields for circular flows in the horizontal planes, with angular momentum $L_z = \pm 1$. They would not be sufficient for an ambitious attempt to construct realistic models, but they may be enough for our main purpose, which is to establish the versatility of the action principle.

Eq.s(2.4) and (2.6) give

$$\vec{v} = \kappa\vec{w} - (\kappa^2+1)\vec{\nabla}\Phi = \omega(-y, x, 0), \quad \omega := \frac{1}{r^2}\left(\frac{\kappa b}{\rho} + a(\kappa^2+1)\right). \quad (2.7)$$

The most general harmonic vector field of this form is a series of spherical functions with higher angular momenta. There is evidence of higher angular momenta in the flow velocities of Venus, Pluto and, most notably, Saturn. In this paper we reduce the series to the simplest terms, the normalized inverse density taking the form

$$\frac{1}{\rho} = 1 + NR + \eta r^2 + \nu\frac{r^2}{R^3}, \qquad N > 0, \qquad \eta > 0, \qquad (2.8)$$

with constant coefficients N, η, ν. The first two terms have $\ell = -1$, the others $\ell = 1$. The introduction of a non-integrable representation of the rotation group ($\ell = -1$) should be noted. Higher harmonics are needed in the case of Saturn, since this planet shows a distinct, hexagonal flow pattern.

A non-zero value of the last term in (2.8) gives rise to a central hole, shaped like a donut, near the center of the planet; see Fig.(2.3). A very small value of ν results in a very small hole; it may serve as a regularizing device, but it is hardly relevant for the evaluation of the shape of the surface. We adopt the expression (2.8), with $\nu = 0$, as a plausible first approximation to the density profiles of Earth and Mars and, very tentatively, to those of the other planets.

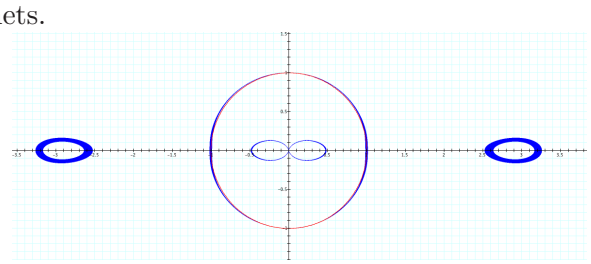


Fig.2.3. The effect of including the last term in (2.8); a hole appears at the center, as well as a ring.

This is our simplest model planet. It is intended, in the first place, to serve as a model for Earth and Mars and perhaps for Uranus and Neptune. A complete calculation would need a general harmonic expansion for the flow vector field $\vec{v}$.

Regularity at the poles

Take the polar radius to be the unit of length and let the central density be the unit of density. Both N and η have to be positive, as we shall see.

Let us begin with a star that is spherically symmetric ($\eta = 0$) with polar radius 1 and density ratio

$$\frac{\rho_{center}}{\rho_{pole}} = N + 1;$$

then we allow for a modest violation of spherical symmetry by increasing the parameter η from zero.

The equation for the surface takes the form,

$$\frac{1}{2r^2}\Big(b^2(1 + NR + \eta r^2)^2 - a^2(1 + \kappa^2)\Big) + \frac{GM}{R} = \text{constant}.$$

In order to avoid getting a dip at the poles (from the denominator r^2) we must have

$$a^2(1 + \kappa^2) = b^2(N + 1)^2. \tag{2.9}$$

Then the equation takes the form

$$\frac{b^2}{2r^2}\Big((1 + NR + \eta r^2)^2 - (N + 1)^2\Big) + \frac{GM}{R} = \text{constant}. \tag{2.10}$$

With this constraint the angular velocity - see Eq.(2.7) - is

$$\omega = \frac{\kappa b}{r^2}(\frac{1}{\rho} \pm \frac{N + 1}{c}), \qquad c := \frac{\kappa}{\sqrt{1 + \kappa^2}}. \tag{2.11}$$

The visible angular velocity at the equator is

$$\omega_{eq.} = \kappa b\left(1 + N + \eta \pm \frac{N + 1}{c}\right). \tag{2.12}$$

The mass transport velocity can change sign within the star.

The Earth has a small equatorial bulge. To estimate the visible angular velocity we approximate η by zero. With these approximations we obtain, using the lower sign, for the visible angular velocity of Earth, the value

$$\omega = b\frac{N+1}{2\kappa} = \frac{a}{2}. \tag{2.13}$$

Finally the shape is determined by

$$f(R,r) := \frac{(1+NR+\eta r^2)^2-(N+1)^2}{r^2} + \frac{\xi}{R} = \text{constant}, \tag{2.14}$$

where ξ is the constant

$$\xi = \frac{2GM}{b^2}. \tag{2.15}$$

Solutions of (2.14) extend to very large R only if there is an effective cancelation between the terms of highest power, in $NR + \eta r^2$. If ηN is positive there can be no cancellation, at any azimuth; hence all the solutions are compact when we pose

$$\eta N \geq 0.$$

Overview of results

The two parameters N and ξ form a Euclidean 2-space with the latter as abscissa; it divides into a lower region (roughly $N < 1$) where the planets have a ring, and a complementary upper region where they do not, separated by a "ring-no-ring" boundary. See Fig. 2.4, where three versions of this dividing line are shown, for $\epsilon = 1/300$ and with $\eta = .01$, $.05$ and $.1$ from high to low. In the same figure we have shown nearly vertical lines of dots, a "trajectory" for each of four planets. The coordinates of the dots on each of the planetary trajectories give a near-perfect fit to the measured ellipsoid of the respective planet.

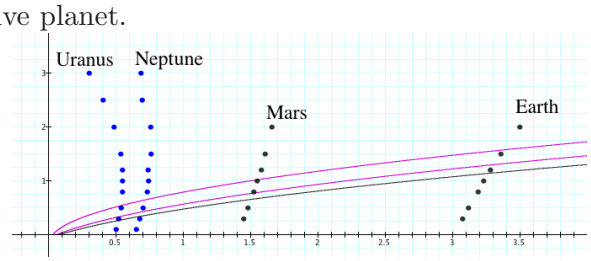


Fig.2.4. The abscissa is the parameter ξ and the ordinate is N.

Fig.2.4 shows the result of calculations as points in the plane of the parameters ξ and N. The near vertical lines connecting dots consist of points that give perfect fits to the shape of the respective planet, without rings on the upper part. Results of the calculation are tabulated in the Appendix.

It may be permissible to think of this diagram as an **evolution diagram**, each planet evolving upwards towards a state of greater compression, and loosing its rings as it crosses the ring-no-ring dividing line. Earth lost its rings long ago; it is to be placed on the upper part of its trajectory, well above the $\eta = .1$ line.

Increasing N means higher compression at the center; Earth may have N as high as 2 while Mars is less compressed and may have $N = 1$ or less. Since planets are likely to become more compressed over time we expect planets to evolve upwards. This is in accord with speculations that Mars may have had a ring in the evolutionary recent past. Uranus and Neptune still have rings and must have $N \leq .3$ if our model is applicable to them.

The rings will be discussed below. Other aspects of the model, including the equation of state, will be taken up in the connection with the gaseous planets, in Section V.

XI.3. Earth and Mars

The polar radius is our unit of length and remains fixed at unity. The equatorial radius is determined by a 4'th order algebraic equation that is obtained from (3.9) by setting $r = R$. We are mostly interested in shapes that are almost spherical (leaving aside the planetary rings for the moment), with a small equatorial bulge. To find surfaces that include a point on the equator with radius $s = 1 + \epsilon$ we write (2.14) in the form

$$B(R, r) := f(R, r) - f(s, s) = 0, \tag{3.1}$$

Note that $f(s, s)$ is a constant. The equatorial radius is a zero of the function $B(r) := B(r, r)$ and after division by $r - s$ this equation reduces to a cubic. For planets without rings this cubic does not have positive roots. If a ring is about to disappear at a distanse s' from the center there will be a double zero at $r = s'$.

The measurable parameter ϵ has replaced the value of the function f. However, there are still 3 parameters left, N, η and ξ, and it is difficult to survey all possibilities. We shall try to find our way around this difficulty as we look at individual planets.

If Earth is an ellipsoid with eccentricity ϵ and the polar radius is normalized to unity, then the shape is

$$R - 1 = \epsilon \sin^2 \theta; \qquad R_{eq} = 1 + \epsilon,$$

with $\epsilon = .00335$. With $s = 1.00335$, the locus $B(R, r) = 0$ passes through the equator at $R = r = s$, and through the pole. Fittings of shapes are relative to this ellipsoid, with the appropriate value of ϵ.

The quality of the fits, examples

Very good fits to the ellipsoid are achieved with $N = 2$ and both of the following $\eta = .092, \quad \xi = 3, \quad$ and $\quad \eta = .1, \quad \xi = 3.5$. The value $N = 2$ was suggested by the measured density profile shown below, in Fig.3.5.

We tried $\eta = .1$ and $N = 2$, leaving only ξ to be varied. The locus is a curve that, at a small scale, resembles the geoid, the fit is perfect at the pole and at the equator. We examined the error at nine intermediate azimuths and found that a perfect fit would require ξ to vary from 0 to 3. But if we fixed $\xi = 3$ the relative error was never larger than 10^{-4}.

The conclusion is that the identification of the planetary shape with a locus of C through the pole and the equator appears to be natural and that the precise determination of the parameters applicable to each planet requires efficient use of more data. In other words, there still remains considerable flexibility to be used as more data is taken into account.

Density profile, range of N

Our own planet is unique among the planets in that the density profile has been reliably estimated, see Fig.3.1. A good fit to the central core is not possible since both N and η must be positive. This can be understood since the constitution of the earth is far from uniform; the model assumption that the interior is a single phase is an over-simplification. A fair approximation to the observed density suggests that N lie in the interval

$$1.5 < N < 2.5 \qquad \text{(Earth)} \tag{3.2}$$

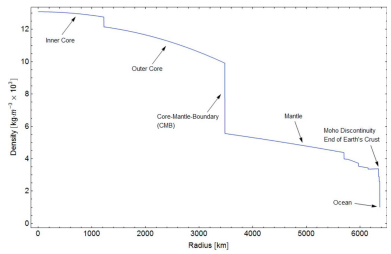


Fig.3.1. Observationally estimated density profile of Earth.

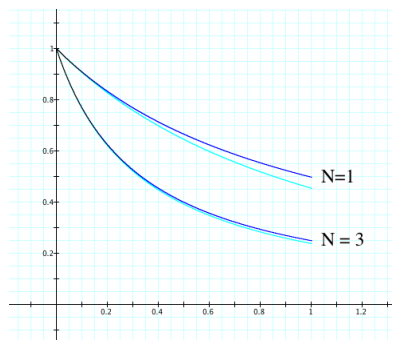


Fig.3.2. Model equatorial density profile of Earth, for 2 values of the principal parameter N. Heavy lines $\eta = .01$, lighter lines, $\eta = .2$.

Rings, or not

Random sampling of the parameters of the theoretical configurations reveal that the expected, nearly spherical shape of the body is not always realized. For example, in the case that $\eta = .1, N = 1.2$ we get a good approximation to the ellipsoidal shape of the Earth with $\xi = 3.3$. But if the value of ξ is increased to 3.525, then a planetary ring appears, as shown in Fig.3.3. We have crossed the ring-no-ring divider in Fig.2.4, moving horizontally towards the right. For still larger values of ξ the ring eventually dwarfs the planet.

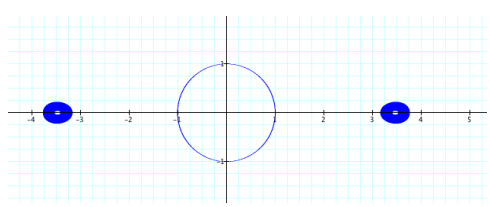
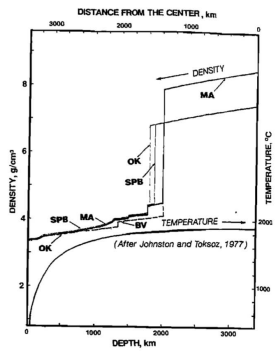


Fig.3.3. Ring around the Earth; about to disappear. Parameters $N = 1.2, \xi = 3.525$.

Nothing in our model relates to the hemispheric asymmetry of Mars; that is, the depression of the northern hemisphere. But we do know the bulge ratio, $\epsilon = 1/135$, and elaborate models of the planet suggest a plausible density profile, shown in Fig. 4.1. We obtain excellent fits to the ellipsoid from $N = .7, \xi = 1.5$ to $N = 3, \xi = 1.75$.



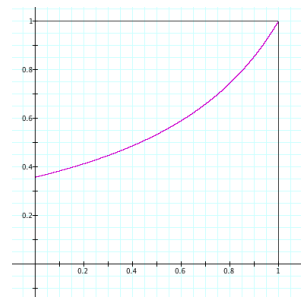


Fig.3.4. Density profile of Mars, with a very thick mantle and a metallic core.

Fig.3.5. Model density profile for Mars from Eq.(2.8), with $N = 1.7$.

The idea of a ring around Earth can not be entertained, but the planet Mars is another matter. The trajectory for Mars was calculated with the equatorial bulge ratio $\epsilon = 1/135$. We fixed the value of η at .1 as for earth and for each of a sequence of values of N we searched for the value of ξ that gives the best approximation to the idealized shape of planet Mars. The lowest value of N for which such a fit exists is $N = .8$.

It is notorious that Mars shows clear evidence of having once been furrowed with large gulleys by the action of water (on the southern hemisphere). It has been widely interpreted in terms of a cataclysmic event, eons past. Our calculations suggest that the value of N was once lower, that a ring actually did exist around Mars, and that the ring, consisting mostly of water or ice, disappeared as a result of the evolutionary increase in N. The alternative, that the ring may have fallen as the result of a passage very close to Earth, is less appealing, since we regard the ring as natural and expect it to resume its original shape after a shock. If it did not, then it means that the property N has evolved and that the equations of motion no longer support the ring.

XI.4. Neptune and Uranus. Venus and Pluto

The shapes of Uranus and Neptune are quoted in the literature but actually they are poorly known (Bertka and Fei 1990). If we treat them as close analogues of Earth and Mars, then they would appear far to the left in Fig.2.4, as shown. The progression of values of the parameter N from Neptune to Earth suggests increasing compression.

Pluto is far out among the outer planet but its small mass is a more relevant parameter (Hellled, Anderson and Schubert 2010). Venus has a complicated structure and winds that are not parallel to the equatorial plane. Both are essentially spherical but Venus has a small bulge that is believed to be induced by winds. Venus also has a very dense atmosphere with a pressure of about 92 Earth atmospheres. For all these reasons neither Venus nor Pluto should be expected to be suitable objects for this study. Nevertheless, we assigned very small bulges and got good fits to the ellipsoid shape, some of which are recorded in Table 1 and plotted in Fig.4.2.

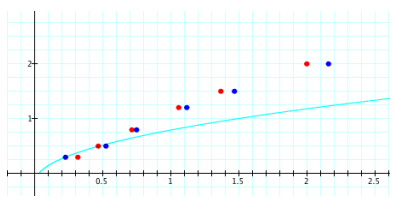


Fig.4.1. This would show why Venus and Pluto do not have rings. While

$\eta \approx .1$ works well for most of the planets, Venus and Pluto require a value closer to .01. In this figure both are plotted with this value of η. All planets except Saturn and Jupiter have "trajectories" determined by a best fit to the ellipsoid.

VI.5. Saturn, Jupiter. The Sun. Haumea. Galaxies.

The search for a simple model for Saturn, initially with no expectation of accounting for anything more than the equatorial bulge revealed that rings are a dominant feature of our model. The existence of rings gives us an additional measurable parameter, the mean radius standing in for the parameters of a complicated ring system. The radius and the width of the

ring can be chosen within wide limits, but the model does not account for the flat ring system that is actually seen. The radius of the ring is closely related to the value of η.

In the case of Saturn attempts to fit the model to ellipsoids with an equatorial bulge ratio of 1:10 failed. We did not persist in this, because: 1. Observation of the gaseous giants does not favor our model of a phase transition at the surface. 2. The surface of a gas sphere is not well defined, experimental data usually refer to isobars. See for example Lindal, Sweetnam and Esleman (1985).

Saturn offers an interesting hexagonal flow pattern around the North Pole. A photograph taken above the North Pole of Saturn, shows a hexagonal flow.

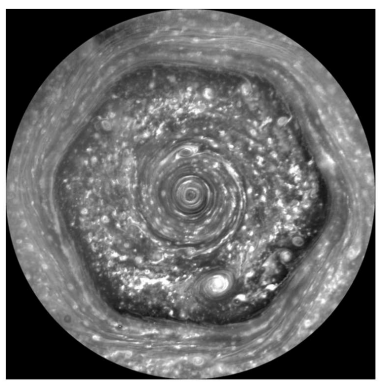


Fig.5.1. Photograph taken by Voyager of the North pole of Saturn.

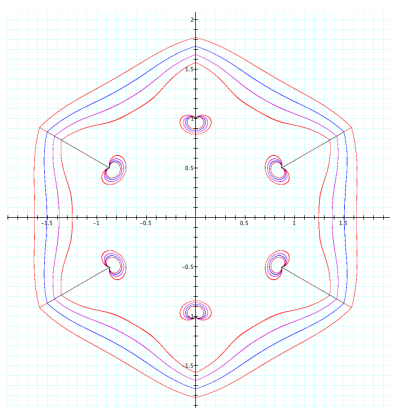


Fig.5.2. Flow lines of the function (6.2) with $a = k = 1, A = .5$.

The photograph reveals a local, inner whorl, with a remarkable similarity to the six whorls that are present in the symmetric model.

A fair mathematical representation of the flow is

$$\Phi = \Im\Big(a \ln z + \frac{A}{1+(kz)^6})\Big), \quad A = \text{constant}. \tag{6.2}$$

The flow lines are shown in Fig. 5.2. The gradient of Φ is

$$\vec{\bigtriangledown}\Phi = \left(a + \frac{A(kr)^6}{D^2}\Big((1+(kr)^{12})\cos(6\phi) + 2(kr)^6\Big)\right)d\phi + \frac{A\sin(6\phi)}{D^2}(1-(kr)^{12})(kr)^5 kdr.$$

For the square we get the surprisingly simple formula

$$(\vec{\bigtriangledown}\Phi)^2 = \frac{a^2}{r^2} + \frac{2a}{r^2}\frac{A(kr)^6}{D^2}\Big((1+(kr)^{12})\cos(6\phi) + 2(kr)^{12}\Big) + \frac{(kA)^2(kr)^{10}}{D^2}.$$

We have calculated only the first order perturbation, then the modified expression for the function $\bar{f}$ in Eq.(2.14) becomes, for some constant α,

$$f(R,r) := \frac{(1+NR+\eta r^2)^2 - (N+1)^2}{r^2}$$

$$+\alpha\frac{r^4}{D^2}\Big((1+(kr)^6)\cos(6\phi) + 2(kr)^6\Big) + \frac{\xi}{R}.$$

The result has been disappointing. It turns out to be possible to produce hexagonal rings, but no further contact with observation was discovered, so far. A more positive result: With the extra term it becomes possible to imitate the ellipsoidal shape.

Jupiter presents some of the same difficulties for analysis, but lacks the interesting hexagonal feature of Saturn. We have not constructed a model for the largest of all the planets.

The Sun is still further from our present objective, and so are galaxies. We present, however, in Fig. 5.3, an object that recalls, by its flatness, the shape of some galaxies; the aspect ratio is about 10^9.

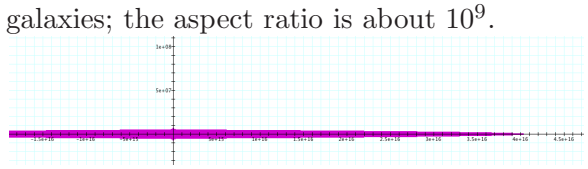


Fig.5.3. A shape produced by the model, suggesting an application to galaxies. (The full figure extends equally in both directions.)

Finally, here is a portrait of one of the smallest object in the solar system, Haumea is a small moon or mini-planet in the outer Kuiper belt, remarkable for its odd shape.

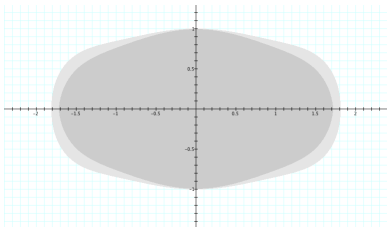


Fig. 5.4. A model likeness to Haumea.

And here is the real Haumea.

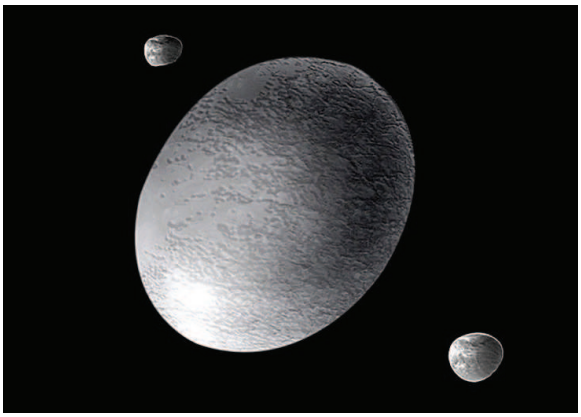


Fig.5.5. A photograph of Haumea taken by Voyager. Not allowed by arXiv.

XI.6. Summary and conclusions

The most significant result of this paper is that an action for hydrodynamics actually provides an effective approach to real problems. The discovery that planetary rings are natural within the formalism is a real surprise and source of encouragement. Rather than a need to invent a special historical event sequence for each ring system, we can now expect that they will emerge

naturally from solutions of the equations of motion. That goes a long way towards explaining their beautiful shapes and their amazing stability.

This paper is to be regarded as a first step, to be followed by much more detailed calculations. The equatorial bulge of a planet changes the gravitational field and this affects the calculations, though often to a minor degree. For each of the objects in the solar system it is important to consider other complications that have hardly been mentioned in this paper, including the following.

Replace the fixed gravitational field by the solution of Poisson's equation. Include more terms in the harmonic expansion of the density, in the hope of creating more elaborate ring systems. The atmosphere was treated as empty, that leaves room for improvement. For the gaseous planets another model, without a surface discontinuity, is indicated. The effect of magnetic fields and radiation must be included.

All these complications have already been taken into account in the literature; we think that the calculations should be repeated within the framework of the action principle, where the energy is conserved and where the number of free parameters is very much reduced.

That should result in a greatly increased predictive power and, perhaps, a better understanding of the fascinating planetary rings. A separate problem that can be approached in the same spirit: the shape of galaxies.

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