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Action Principle for Hydrodynamics and Thermodynamics, including general, rotational flows

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ABSTRACT

This paper presents an action principle for hydrodynamics and thermodynamics that includes general, rotational flows, thus responding to a challenge that is more that 100 years old. It has been lifted to the relativistic context and it can be used to provide a suitable source of rotating matter for Einstein's equation, finally overcoming another challenge of long standing, the problem presented by the Bianchi identity.

The new theory is a combination of Eulerian and Lagrangian hydrodynamics, with an extension to thermodynamics. In the first place it is an action principle for adiabatic systems, including the usual conservation laws as well as the Gibbsean variational principle. But it also provides a framework within which dissipation can be introduced in the usual way, by adding a viscosity term to the momentum equation, one of the Euler-Lagrange equations. The rate of the resulting energy dissipation is determined by the equations of motion. It is an ideal framework for the description of quasi-static processes. It is a major development of the Navier-Stokes-Fourier approach, the principal advantage being a hamiltonian structure with a natural concept of energy as a first integral of the motion.

Two velocity fields are needed, one the gradient of a scalar potential, the other the time derivative of a vector field (vector potential). Variation of the scalar potential gives the equation of continuity and variation of the vector potential yields the momentum equation. The dual nature of the velocity field has a profound effect on the structure of the energy momentum tensor which, in turn, affects the coupling to the gravitational field, as will be discussed elsewhere (Fronsdal 2014c).

This paper is dedicated to the memory of my friend Volodya Kadyshevsky.

I. Introduction

Many branches of theoretical physics have found their most powerful formulation as action principles. An action principle was known for Eulerian hydrodynamics, but it is limited to potential flows and its use is restricted. An alternative version of hydrodynamics, the 'Lagrangian' formulation, also has an action principle formulation, but it does not incorporate the equation of continuity. The purpose of this paper is to formulate an action principle for hydrodynamics, with an extension to thermodynamics, that includes the equation of continuity and that encompasses general velocity fields.

The principal conclusion of this paper is that, to describe general flows, it is necessary to combine two versions of hydrodynamics that, rather than being alternatives are actually complementary. In hindsight, after arriving at this conclusion through an analysis of the Navier-Stokes equation, it is possible to argue much more simply as follows. Consider a cylinder filled with an incompressible fluid, rotating around the central axis. We shall replace all positive constants by 1.

In potential theory the kinetic energy density is part of the Hamiltonian density, h, say, and there is a force in the direction of the gradient of $-h$. Potential flow has $-v^2 = -1/r^2$, where r is the distance from the axis. The gradient of $-v^2$ is $1/r^3$, positive; the centrifugal force is outwards as it should be. But if we should try to describe flow of the solid-body type, when $-v^2 = -r^2$, then the force would be $-r$, pointing inwards, which would be absurd. We are saved from committing this error because this latter kind of flow is not potential flow.

In the Lagrangian' version of hydrodynamics the Lagrangian density is $v^2 - h$ and the centrifugal force is the gradient of this quantity, more precisely the gradient of the kinetic term. Here the situation is reversed. In the case of solid-body rotation $v^2 = r^2$ and the force is r, directed outwards as it must be. But nothing would warn against applying this theory to potential flow, and that would then give a force directed inwards, which is absurd. Conclusion: if we wish to describe rotational motion in general, then we need to have a term v^2 in the kinetic part of the action to describe solid-body motion, as well a term u^2 in the hamiltonian to describe potential flow. That is, we need two velocity fields.

The Navier-Stokes equation has maintained a dominant position in hydrodynamics for more than 100 years. It has known numerous successes and no real failures, but see Brenner (2013) and especially the review by Martin (2010). It suffers, nevertheless, from an inherent incompatibility with the energy concept. This is perhaps acceptable, since it was designed to describe processes in which energy is dissipated, but it implies a lack of completeness for which most workers have felt a need to compensate. It is therefore usually supplemented by an 'energy equation'. But the choice of an expression for this 'energy' is seldom canonical and cannot be fully justified. Dissatisfaction with this approach is occasionally expressed in the literature, as in this example (Khalatnikov (1965). After listing a "complete system of hydrodynamic equations" (equations 2-5) he presents one more, "the energy conservation law $\partial E/\partial t + \mathrm{div}Q = 0$", and then he says: "It is necessary to choose the unknown terms in Eq.s (2-5) in such a way that this last equation be automatically satisfied."

To formulate an action principle one begins by choosing the dynamical variables.

The principal variable is a velocity field but, in order that Euler-Lagrange equations give a differential equation for the velocity, the velocity has to be represented as a derivative of a basic, dynamical variable. In one case it is a space derivative, $\vec{v} = -\vec{\bigtriangledown}\Phi$, in the other case it is a time derivative, $\vec{u} = d\vec{X}/dt$.

In the case of potential flows of a homogeneous fluid the velocity field is the gradient of a scalar potential. In this case the Eulerian point of view is natural and convenient; the density and the scalar potential form a canonically conjugate pair of dynamical variables. Variation of the scalar velocity potential yields the equation of continuity. In general, the variational approach requires a vector potential as well, such as appears in the alternative, Lagrangian formulation, where the velocity is represented as the time derivative of a vector field (a vector potential). A variational formulation is well known, see for example Landau and Lifshitz (1958) but it does not incorporate an equation of continuity. In the 'Lagrangian' approach $\vec{v}, \vec{X}$ is a pair of conjugate variables. The density does not have a conjugate momentum and there is no equation of continuity.

It is a well known and remarkable fact that some phenomena in fluid mechanics are characterized by two kinds of flow. The vortices that are seen in the wake of ships are created locally; the angular momentum decreases as the inverse square of the distance from the center; this is characteristic of potential flow. In experiments involving fluids confined inside rotating cylinders (Couette flow) it is observed that motion generated by rotating the enveloping vessel tends to create a flow pattern that is close to that of a solid body, accompanied by local vortices that are described in terms of potential flow.

Here the Navier-Stokes equation comes through with the successful prediction that, in the presence of viscosity, precisely these two kinds of flow are sustainnable.

In their study of flow patterns in super conducting Helium, Landau (1954) and Feynman (1954) describe the small vortices in terms of potential flow ('phonons'), while the alternative, competing flow is referred to as 'motion of the system as a whole'; or solid-body flow. This point of view is still dominant in the more recent literature, see for example the review by Fetter (2009). But no independent field theoretical degree of freedom was associated with this other kind of flow.

The variational principle for hydrodynamics that is presented here combines potential flows and 'solid-body' flows, just as Navier-Stokes does, but two independent vector fields are needed, one derived from a scalar potential, the other from a vector potential.

The introduction of a pair of velocity fields is by no means new. Please see the quoted review by Martin (2010) as well as earlier reviews quoted therein.

We study the application of Navier-Stokes to a simple system, in order to learn how it deals with physics that has not been described within a variational approach. It turns out that, in the chosen context, this inspires a resurrection of the 'Lagrangian' formulation of hydrodynamics, where the velocity is represented as a time derivative, $\vec{v} = d\vec{x}/dt$. An action principle for general flows must combine the two classical formulations of hydrodynamics, the Eulerian point of view and the 'Lagrangian' approach.

The emphasis on hydrodynamical concepts is natural, but we wish to emphasize the broader perspective of thermodynamics, where the reliance on the Navier-Stokes equation has been much less rewarding. Fig. 1 is meant to evoke the structure of the

Gibbsean geometric concept of thermodynamics, the space of unconstrained dynamical variables over which the variation is to be carried out and the subspace of extrema that are the physical states. The variation is carried out with the entropy distribution fixed, the choice of entropy determines a well defined Lagrangian and corresponds to one of the subspaces illustrated by the blue lines (approximately parabolas) in the figure. The red line (at the bottom, nearly straight) is the collection of equilibrium states, one for each adiabatic system. 'Non-equilibrium thermodynamics' refers to the problematics that is presented when, initially, only the states of equilibrium are known. We take the position that adiabatic thermodynamics is a developed discipline and that this gives a much wider platform from which to launch a study of dissipation. A change of state that involves viscosity and other forms of dissipation is best described as a quasi-static evolution along a sequence of adiabatic equilibria; it is evident that such studies must be built on a coherent account of the adiabatic systems themselves, not just their equilibria (Prigogine 1965). An adiabatic system gives a meaning to the concept of equilibrium, a family of adiabatic systems is needed to describe dissipation.
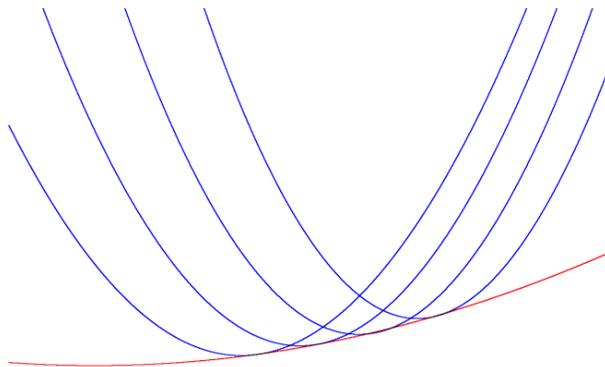

Fig.1. Geometric presentation of Gibbs' thermodynamics.

The simplest non-adiabatic evolution is a sequence of equilibria, of a family of adiabatic systems parameterized by the entropy. The very concept of equilibrium only makes sense in the dynamical context within which it is the equilibrium configuration! It is evidently useful to have an efficient formulation of general adiabatic systems and, at best, a formulation in the form of action principles.

The dominant strategy that has been pursued under the banner of nonequilibrium thermodynamics is based on a collection of conservation laws, suitably modified to account for dissipation. The most important are the equation of continuity (mass conservation) and the Navier-Stokes equation (in the absence of viscosity: momentum conservation). The Navier-Stokes equation has a long and impressive history, including numerous successful applications. As we look for a variational principle we must try to incorporate as many properties of Navier-Stokes as we can. That is the strategy that characterizes this paper.

The original aim of this work was to find appropriate matter sources for the dynamical metric field of General Relativity (Fronsdal 2007, 2014c) and the present work

represents a step in that direction. This report sticks to the non relativistic context, classical hydrodynamics and thermodynamics. Nevertheless one remark should be made.

The essential step that lifts classical hydrodynamics to the relativistic context is commonly regarded to be the promotion of the 3-vector velocity to a timelike, 4-vector field. This is natural in the case of a gradient vector field but not when the velocity is presented as a time derivative of a 3-vector field. In that case the relativistic velocity is the dual of an exact 3-form. This make a very big difference in the structure of the energy momentum tensor; see the last paragraph of this paper. Applications to electromagnetism are not discussed here. Implications for General Relativity are described in a recent paper (Fronsdal 2014c).

Summary of the paper

Section II presents the well known variational formulation of potential flows in hydrodynamics, and a short account of the connection to thermodynamics. The next section describes the strategy that was chosen to look for an action principle of wider generality: to focus on the special case of cylindrical Couette flow, with an unusual emphasis on laminar flow, in this simplest context in which non-potential flows are inevitable. Section IV reviews the classical application of the Navier-Stokes equation to this system and contrasts it with the failure of the potential theory, to discover the tight spot, and the remedy.

Section V presents the proposed solution to the problem, a solution that is both simple and natural. It combines Eulerian and 'Lagrangian' hydrodynamics to describe both potential and rotational flows.

Section VI is a discussion of the relative merits of the variational principle and the Navier-Stokes equation in the context of adiabatic dynamics where both are applicable. The main issue is the role of energy. The extension that includes viscosity is natural and standard. It is suggested that the inclusion of a new degree of freedom may affect stability studies. This section is followed by a brief study of plane Couette flow, where the relationship between the two theories turns out to be very different. The variational approach has the greater predictive power, in this case, but experimental confirmation must wait for more accurate temperature measurements.

Section VIII is a very brief summary of the main conclusions and the possible applications.20

II. The Fetter-Walecka action principle

A variational formulation of simple hydrodynamics may be found in a book by Fetter and Walecka (1980). The action is

$$A = \int dt \int d^3x \, \mathcal{L}, \quad \mathcal{L} = \rho(\dot{\Phi} - \vec{v}^{\,2}/2) - W(\rho), \tag{2.1}$$

with the definition

$$\vec{v} = -\mathrm{grad}\,\Phi. \tag{2.2}$$

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That is a strong restriction on the velocity field. The potential W is related to the pressure, as will be shown.

The equations of motion are the Euler-Lagrange equations derived by variation of the two scalar fields. Variation of Φ gives the equation of continuity,

$$\dot{\rho} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{2.3}$$

and variation of ρ,

$$\dot{\Phi} - \vec{v}^2/2 = \frac{\partial W}{\partial \rho}, \tag{2.4}$$

or

$$\rho\dot{\vec{v}} + \rho\vec{\nabla}\vec{v}^2/2 = -\operatorname{grad} p, \quad p := \rho\frac{\partial W}{\partial \rho} - W, \tag{2.5}$$

where p is the pressure. The two terms on the left side can be combined,

$$\dot{\vec{v}} + \vec{\nabla}\vec{v}^2/2 = \frac{D\vec{v}}{Dt} := \dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla})\,\vec{v}. \tag{2.6}$$

That the substantial derivative $D\vec{v}/Dt$ appears here, as it should, comes about because of the identity

$$\vec{\nabla}\,\vec{v}^2/2 = (\vec{v} \cdot \vec{\nabla})\,\vec{v},$$

which is true by virtue of (2.2). Eq. (2.5) is the Bernoulli equation, written in the form in which it is most easily compared to the Navier-Stokes equation. It is the gradient of the 'integrated Bernoulli equation' (2.4).

To gain some impression of the significance of this formulation see the review by Fetter (2009), where the quantization of vortices (Onsager 19??) is expressed in terms of the one-valuedness of $\exp(i\Phi/\hbar$ (constant factors omitted).

Onsager ref.

Thermodynamics

This action principle can be combined with Gibbs thermodynamic principle of minimum energy (Gibbs 1878). Applied to a one component thermodynamic system the lagrangian density takes the form

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2) - f(\rho, T) - sT. \tag{2.7}$$

Here f is the free energy density and $s = \rho S$ is the entropy density. Variation of T (the temperature) gives the adiabatic equation that permits the elimination of the temperature in favor of the specific entropy density S, assumed uniform. Variation of Φ gives the equation of continuity and variation of the density gives

$$\dot{\Phi} - \vec{v}^2/2 = q, \tag{2.8}$$

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where q is the chemical potential density. The chemical potential can be expressed in terms of the pressure and the entropy, which leads to the Bernoulli equation or, in the static case, to the hydrostatic equation. But this requires knowledge of the entropy. In the case of an ideal gas q can expressed in terms of the temperature and we obtain in that case

$$\dot{\Phi} - \vec{v}^{\,2}/2 = C_V T. \tag{2.9}$$

This form of the equation is, in our opinion, much to be preferred, for it requires no knowledge of the entropy. The comparison of theory and experiments and, consequently, the interpretation, would be greatly facilitated if it were possible to make more accurate measurements of the temperature profile.

Up to this point in our work (Fronsdal 2014a, 2014b) the attention has been focused, deliberately, on the special case when the flow velocity is a gradient, the curl being zero,

$$\vec{v} = -\vec{\nabla}\Phi, \quad \rightarrow \quad \vec{\nabla} \wedge \vec{v} = 0.$$

Although it is unduly restrictive, this limitation was accepted as an apparently necessary condition to formulate hydrodynamics (and thermodynamiccs) as a lagrangian field theory. Many problems in thermodynamics and hydrodynamics involve no flow and in many others the flow is irrotational. But a relativistic lagrangian is needed for General Relativity and this is one of many applications where the restriction to potential flows must be overome. To discover how to achieve greater generality we shall now turn our attention to a simple system where a generalization is obviously and urgently needed.
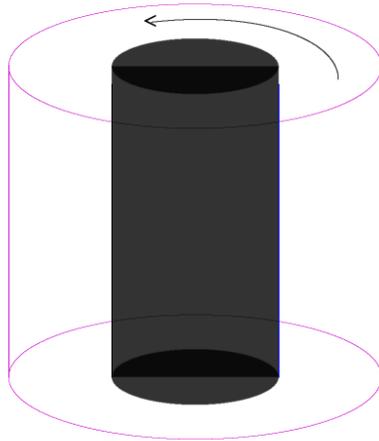

Fig.2. Cylindrical Couette flow is the steady, horizontal, rotational flow between two concentric cylinders.

III. Couette flow, potential flow

We shall consider a situation that is effectively 2-dimensional because of translational symmetry, when nothing depends on a vertical coordinate z and the flow is parallell to the horizontal x, y plane. Couette flow is the flow of a fluid between a pair

of concentric cylinders that can be rotated around a common (vertical) axis. With both cylinders at rest we postulate an initial state in which the space bounded by the two cylinders is filled with a fluid at rest, with all variables time independent, and uniform. The effect of gravity will be neglected. The cylinders are long enough that end effects can be neglected as well.*

We begin to rotate the inner cylinder. For this to have any effect on the fluid we need to postulate a degree of adherence of the liquid to the surface of the inner cylinder. We suppose that a state of stationary rotation is approached asymptotically and that there is no loss of energy at the wall in the limit. This is summed up by the no-slip boundary condition

$$\vec{v}|_{r=r_0} = \vec{v}|_{\text{inner boundary}} = \omega_0(-y, x, 0).$$

The angular velocity ω_0 is that of the cylinder.

The no-skip boundary condition has been widely applied, and because we wish to compare our approach to the traditional one it serves our purpose to do the same. See Brenner (2011), Priezjev and Troian (2005), Dukowicz, Stephen, Price and Lipscomb (2010), Goldstein, Handler and Sirovich (1993).

The rotational axis is the z axis. The coordinates are inertial and Cartesian. No boundary conditions are imposed on the velocity at the outer wall, so far.

We propose to treat this system, with fixed boundary conditions, as an adiabatic system, and the stationary solution as its equilibrium configuration. We begin with the Fetter-Walecka Lagrangian density

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2) - f - sT. \tag{3.1}$$

The problem is essentially 2-dimensional and r is the cylindrical, or polar radial coordinate. When all the variables are time independent the Euler-Lagrange equations reduce to

$$\text{div}(\rho\vec{v}) = 0, \quad \dot{\Phi} - \vec{v}^2/2 = q, \tag{3.2}$$

where q is the chemical potential and $\dot{\Phi}$ is a constant. In the case of an ideal gas ,

$$\dot{\Phi} - \vec{v}^2/2 = C_V T. \tag{3.3}$$

(See Eq.s(2.8) and (2.9).) If we assume that the specific entropy is uniform, then this last equation can be transformed to the more familiar Bernoulli equation,

$$\dot{\vec{v}} + \rho\vec{\nabla}(\vec{v}^2/2) = -\vec{\nabla}p. \tag{3.4}$$

For this stationary state the flow lines are circles, $r^2 = x^2 + y^2 = \text{constant}$,

$$\vec{v} = \omega(r)(-y, x, 0).$$

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* According to a Merriam Webster web page, the term Couette flow is derived from "French couette machine bearing, literally, feather bed, from Old French coute, cuilte quilt, mattress." To make up for this slight we cite several of Couette's pioneering papers (Couette 1887,1888,1889,1890).

It is a field with vanishing curl only if

$$\omega(r) = ar^{-2}, \quad a = \text{constant},$$

and even then it is not, in the strict sense, a gradient. Although

$$\vec{v} = a \operatorname{grad}\theta, \quad \theta := \arctan\frac{y}{x}, \tag{3.5}$$

the scalar θ is not one-valued, though the vector field is. Thus

$$\vec{v} = a\frac{(-y, x, 0)}{r^2} = -\vec{\nabla}\Phi, \quad \Phi = a\theta, \quad a = \text{constant},$$

one extends the approach to include special rotations, with $\vec{\nabla} \wedge \vec{v} = 0$ away from the origin only (though ill defined at that one point). In the present case the origin is outside the vessel and the curl is zero at all points of the fluid. It is customary and convenient to encompass this situation in the concept "potential flow".

This is the only possible horizontal, circulating, potential flow. The angular velocity is greater at smaller radius; it is therefore likely that it is driven by rotating the inner cylinder and that it settles according to the no slip condition. There is no opportunity to adjust the outer boundary conditions; either the fluid slips there or else the angular speed of the outer cylinder must be adjusted in accordance with $\omega(r)r^2 = a$.

The divergence of $\vec{v}$ (for any choice of $\omega(r)$) is zero, so the equation of continuity reduces to $\vec{v} \cdot \vec{\nabla}\rho = 0$, requiring ρ to depend on r and z only. From now on it is taken for granted that ρ is a function of r only.

This type of stationary motion has been observed, see for example Couette (1887), Joseph and Renardy (1985), but the principal goal of most experiments has been to measure the onset of turbulence as, with increasing rotational speeds, the laminar flow breaks down . Our interest shall be focused, instead, on the laminar flow at low angular speeds.

The principal equation of motion, Eq. (3.3) reduces, in the case that the velocity takes the form (3.5), in the case of an ideal gas, to

$$\frac{a^2}{2r^2} = C - C_V T, \quad C = \text{constant}. \tag{3.6}$$

It appears that this temperature lapse has not been measured. The experiment would be difficult to interpret because of the suspected heat flow caused by the friction, within the liquid and between the liquid and the wall. The equation is consistent with the interpretation of the fluid as a collection of classical particles, and in full agreement with the Navier-Stokes equation. * As is seen from Eq. (3.4), the kinetic energy acts as an effective potential and gives rise to a repulsive, radial force, balanced by the negative of the pressure gradient.

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* It is common, in the case of liquids, to treat them as incompressible. In this case it is justified to treat the pressure gradient as unknown, to be determined by the need to get reasonable solutions to the equations. But of course this puts severe limits on the predictive power of the theory.
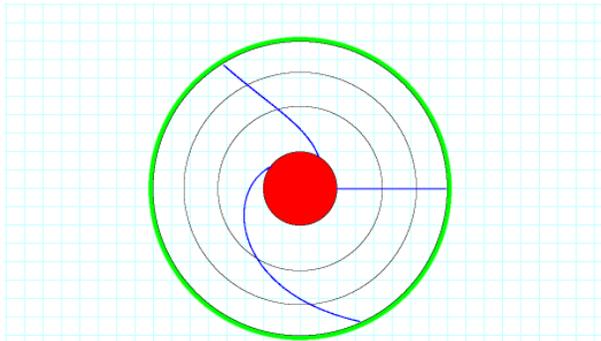
Fig. 3. Snap shots of Couette flow. The horizontal line shows the position of a set of particles at $t = 0$; the two curved lines the same particles at later times. The inner cylinder is rotating clockwise, dragging the fluids along with it.

'Solid-body' flow

Consider next the complimentary experiment in which there is friction at the outer cylinder only, making it the driver of the motion. The problem is the same, except for different boundary conditions,

$$\vec{v}|_{r=r_1} = \vec{v}|_{\text{outer boundary}} = \omega_1(-y, x).$$

Our initial study of potential motion revealed only one solution, and here its application is anti-intuitive since it has the angular velocity increasing towards the center, away from the driving wall. For the general problem, in which the two cylinders move independently, we have two independent boundary conditions but a one dimensional space of potential flows. Clearly we have come to a situation where we cannot limit our attention to gradient velocity fields. Still, we are not willing to concede that this is the limit of the action principle approach to hydrodynamics.

It is reported that the motion of a liquid, even a rarified gas, driven by the rotation of the outer cylinder, tends towards the motion of a solid body. See for example Andereck, Liu and Swinney (1986), de Socio, Ianiro and Marino (2000). As the speed is increased, various instabilities set in, but at present our only concern is to understand the laminar motion observed at low speeds.

The motion of a rigid body is characterized by the velocity field

$$\vec{v} = b(-y, x, 0), \quad b \text{ constant},$$

and this is not a gradient. If we analyze this flow in the same manner that we did potential flow; that is, if we simply use the above velocity field in the field equations, then we obtain similar results; in particular, instead of (3.6),

$$\frac{b^2}{2} r^2 = C - C_V T.$$

(Note that to get this unphysical result we applied the equations of the potential theory beyond their domain of validity.) Here the left side has the wrong sign, the force that

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it implies is attractive. This error in sign is a problem that frustrates all efforts, until the only possible solution pops into view.

We have to escape from under the restriction to gradient velocities and a generalization of the action principle is required. It is natural to ask how Navier-Stokes handles handles this situation.

IV. Navier-Stokes

The standard treatment of non-potential flows is based on the continuity equation and the Navier-Stokes equation (Navier 1827, Stokes 1843, Navier 1882),

$$\dot{\rho} + \text{div}(\rho \vec{v}) = 0,$$

$$\rho\left(\dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla})\vec{v}\right) = -\vec{\nabla}p - \mu\Delta\vec{v}. \tag{4.1}$$

This allows for flows of both kinds, potential flow and solid body flow. The new elements are three. First of course, the nature of the velocity field is not constrained to be a gradient. In the second place the term $\rho\vec{\nabla}\vec{v}^2/2$ in (3.4) has been replaced by the term $\rho(\vec{v} \cdot \vec{\nabla})\vec{v}$ in (4.1). Finally, there is the viscosity term $\mu\Delta\vec{v}$.

If the coefficient μ (the dynamical viscosity) vanishes, of the second equation only the radial component remains. It is an ordinary differential equation for the functions ω, ρ and p and the system is under-determined. If $\mu \neq 0$, the tangential projection imposes the additional requirement that

$$\Delta\vec{v} = 0. \tag{4.2}$$

This leads to unique solutions for reasonable boundary conditions, either or both cylinders driving. The fact that uniqueness is obtained only when the viscosity is taken into account may seem a little odd at first, since there is no lower limit on the value of the coefficient μ. But actually this is natural since we are considering stationary solutions only; the stationary solutions are those that avoid the dissipation induced by viscosity. And this is the standard treatment. Given the form

$$\vec{v} = \omega(r)(-y, x, 0)$$

of the velocity field (no longer required to be a gradient) within the class of flows under consideration, the general solution of Eq.(4.2) is

$$\vec{v} = \omega(r)(-y, x, 0), \quad \omega(r) = \frac{a}{r^2} + b, \tag{4.3}$$

a and b constants; the two types of flow already considered are the only ones allowed by Eq. (4.2).

The boundary conditions at $= r_0, r_1$ give us

$$\omega(r_0) = a{r_0}^{-2} + b = \omega_0, \quad \omega(r_1) = a{r_1}^{-2} + b = \omega_1,$$

When the 'a'' term dominates we have the highest angular velocity at the inner surface; this is as expected when the inner cylinder is driving. If the 'b'' term dominates we have nearly constant angular velocity, as in the case of a solid body. If both cylinders are rotating, in opposite directions, and both are driving, then $\omega(r)$ will have a change of sign. Explicitly,

$$a = \frac{\omega_0 - \omega_1}{r_0^{-2} - r_1^{-2}},$$

and

$$b = \frac{1}{r_1^2 r_2^2} \frac{r_1^2 \omega_1 - r_0^2 \omega_0}{r_0^{-2} - r_1^{-2}}.$$

The result is that Navier-Stokes, with non zero viscosity, has just one extra solution besides the gradient, allowing it to satisfy no slip boundary conditions for all values of the angular velocities of the two cylinders. It is important that this new solution, with $\vec{v} \propto (-y, x, 0)$, is the same as the static state observed from a rotating reference frame; its existence is required by the relativistic equivalence theorem. And the fact that the Navier-Stokes singles out just two, radically different kinds of flow, and nothing else, is highly significant.

The salient differences between Navier-Stokes and the theory attempted in the preceding subsection are the following. As we said, the term $\vec{\bigtriangledown} v^2/2$ in Eq.(3.3) is replaced by the term $\vec{v} \cdot \vec{\bigtriangledown} \vec{v}$ in (4.1). One verifies that, when the velocity field is a gradient, both expressions are equal, but in the case of solid body rotation there is a change of sign,

$$\vec{\bigtriangledown} \vec{v}^2/2 = b^2 \vec{r} \ \text{ but } \ \vec{v} \cdot \vec{\bigtriangledown} \vec{v} = -b^2 \vec{r}.$$

We need to understand what lies behind this change of sign.

The centrifugal force

In particle physics the dynamical variable is the position of the particle. The 'fictitious' centrifugal force in particle physics can be seen as coming from an effective 'kinematic' potential. The force is $\omega^2 \vec{r}$ directed outwards, and this is $-\vec{\bigtriangledown}[-\omega^2 r^2/2]$, so the effective potential is the negative of the kinetic energy. The unexpected sign comes from the fact that the the origin of this potential is in the term $m\dot{\vec{x}}^2/2$ in the Lagrangian; this term appears with the same sign in the Hamiltonian. Let us say it in another way, the term in question appears with positive sign in the Lagrangian and with the same sign in the Hamiltonian; but its contribution to the equation of motion is opposite in sign from that of a normal potential. We have seen that our potential model when used outside its domain of validity, gives the wrong sign in the equation of motion.
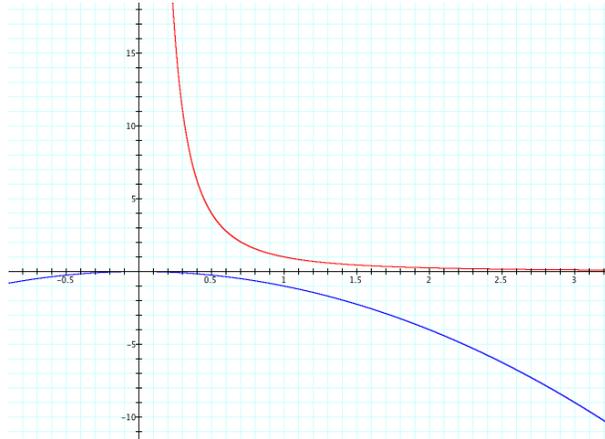
Fig. 4. Centrifugal potentials. The abscissa is the distance from the axis of rotation. The upper curve is the potential associated with a gradient velocity, a^2/r^2; it appears as a positive contribution to the Hamiltonian and with negative sign in the Lagrangian. The force is the negative of the gradient of the potential, pointing outwards. The lower curve is the negative of the potential associated with solid body rotation. This potential, $b^2 r^2/2$, appears with positive sign in the Hamiltonian, and with the same sign in the Lagrangian; the force is in the direction of the gradient, likewise outwards.

The two classical formulations of hydrodynamics

Fluid mechanics can be understood as a field theory, and there are two versions, some times said to be equivalent (Stanyukovich 1960, Sedov 1971), a Eulerian formulation and an alternative 'Lagrangian' formulation. The principal distinction is the following.

In the Eulerian formulation of fluid mechanics the dynamical variables are the scalar density field and a velocity vector field. This theory becomes a Lagrangian field theory, in the sense of being based on a dynamical action principle, only in the case that the velocity field is the gradient of a velocity potential. Thus formulated, this theory cannot describe solid body rotations because the potential $\rho \vec{v}^2/2$ appears with the negative sign in the Lagrangian. Let us be clear about this: the velocity associated with rotational motion in the x, y plane is $\omega(-y, x, 0)$ and it is not a gradient, but there is an additional obstacle. If we introduce a term $-\vec{v}^2/2$ (or $+\vec{v}^2/2$) in the Lagrangian we get the right sign in the Hamiltonian but the wrong sign in the equations of motion (or *vice versa*).

The 'Lagrangian' version of fluid mechanics handles the centrifugal force correctly, just as particle physics does, but the kinetic energy is in the kinetic part of the Lagrangian. This way one gets the right sign for solid-body motion. This is because the dynamical variables are not ρ and $\vec{v}$ but ρ and a vector field usually denoted $\vec{x}$, satisfying $d\vec{x}/dt = \vec{v}$. To avoid confusion we shall call it $\vec{X}$. The books explain that this vector field may be interpreted as the position of a particle in the fluid; one chooses an initial value and calls on the equations of motion to predict the future trajectory. This explanation tends to obscure the fact that this version of the theory is also a field theory, with dynamical field variables ρ and $\vec{X}$, and

$$\vec{v} := \dot{\vec{X}}.$$

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The velocity is the time derivative of the basic field $\vec{X}$ and the term $\rho\,\vec{v}^2/2$ is now in the kinetic part of the Lagrangian. Nothing appears to prevent the velocity from being potential type, but in that case the centrifugal force shows up with the wrong sign.

The Navier-Stokes equation is based directly on particle mechanics. But the main equation does not involve the gradient of a Hamiltonian and no Hamiltonian exists. The dynamical field is the velocity but the gradient of no kinetic potential enters the calculations. Instead the substantive derivative appears,

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\bigtriangledown})\vec{v}.$$

Let us look once more at the example of rotary motion. The gradient theory describes a motion of the type $\vec{v} = (a/r^2)(-y, x, 0)$, and in this case

$$(\vec{v}\cdot\vec{\bigtriangledown})\vec{v} = \vec{\bigtriangledown}\vec{v}^2/2.$$

But in the case of solid body motion, when $\vec{v} = b(-x, y, 0)$ (a and b are constants), we find that

$$(\vec{v}\cdot\vec{\bigtriangledown})v = -\vec{\bigtriangledown}\vec{v}^2/2.$$

The Navier-Stokes equation is based on particle mechanics and both kinds of motion are treated correctly, but there is no Hamiltonian; this is because the velocity is the primary dynamical variable. To get beyond this stage, to set up a Lagrangian theory of non potential flow, it is necessary to introduce the field $\vec{X}$, as in the 'Lagrangian' version of fluid mechanics. Traditional 'Lagrangian' hydrodynamics is formulated as an action principle but it does not incorporate the equation of continuity; it is not hydrodynamics, a scalar velocity potential is needed as well.

V. Action principle for general flow

With the Lagrangian density that we have used for potential flow as a starting point, we begin by adding a new term,

$$\mathcal{L} = \rho\Big(\dot{\Phi} - \frac{1}{2}(\vec{\bigtriangledown}\Phi)^2 + \frac{1}{2}\dot{\vec{X}}^2\Big) - f - sT. \tag{5.1}$$

The term $\dot{\vec{X}}^2/2$ appears with the sign that is appropriate for solid-body rotational motion.

We digress to show that this Lagrangian has precedents. The potential term $-(\rho/2)(\vec{\bigtriangledown}\Phi)^2/$ may be considered as a part of the free energy density; the new term $(\rho/2)\dot{\vec{X}}^2$ plays exactly the same role as the second term in

$$F' = F - M\Omega. \tag{5.2}$$

where $M = I\Omega$, I the total moment of inertia. This equation appears in the work of Hall and Vinen (1956) (and in Landau and Lifshitz 1955, 1958); F is the free energy,

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including the kinetic energy due to potential flow, M is the total angular momentum and Ω is the angular velocity. This quantity F' is required to be a minimum with respect to variation of Ω, with I fixed, and the resulting equation is the same as when $I\Omega^2/2$ is varied with M held fixed. The context is superfluid Helium but the theory applies just as well to normal fluids. The physical interpretation as well as the mathematical structure is thus precisely the same as that of the above tentative Lagrangian (5.1). This explains the negative sign: F' is not a modified free energy but a Lagrangian! End of digression.

The Hamiltonian density is now

$$\mathcal{H} = \frac{\rho}{2}\dot{\vec{X}}^2 + \frac{\rho}{2}(\vec{\nabla}\Phi)^2 + f + sT. \tag{5.3}$$

This is correct for all flows, most notably for solid body rotation, $\dot{\vec{X}} = b(-y, x, 0)$.

We shall compare this theory with the Navier-Stokes equation, but we first need to add another term to the Lagrangian, the final form being

$$\mathcal{L} = \rho\left(\dot{\Phi} + \frac{1}{2}\dot{\vec{X}}^2 + \kappa\dot{\vec{X}} \cdot \vec{\nabla}\Phi - \frac{1}{2}(\vec{\nabla}\Phi)^2\right) - f - sT. \tag{5.4}$$

Variation of Φ will give the equation of continuity and the term $\kappa\rho\dot{\vec{X}} \cdot \vec{\nabla}\Phi$ is included in order to get a correct expression for the flow. The factor κ is an adjustable, dimensionless parameter. The expression (5.3) for the Hamiltonian is not affected.

Let us list all the Euler - Lagrange equations.

1. Variation of T: the adiabatic relation, as always.

2. Variation of Φ: the continuity equation, now

$$\dot{\rho} + \vec{\nabla} \cdot \left(\rho\kappa\dot{\vec{X}} - \rho\vec{\nabla}\Phi\right) = 0. \tag{5.5}$$

The unique velocity of mass transport is therefore

$$\vec{v} := \kappa\dot{\vec{X}} - \vec{\nabla}\Phi. \tag{5.6}$$

This is what corresponds to the velocity that appears in the Navier-Stokes equation.

3. Variation of the vector field:

$$\frac{\partial}{\partial t}\vec{m} = 0, \quad \vec{m} := \frac{\partial}{\partial t}\left(\rho\dot{\vec{X}} + \rho\kappa\vec{\nabla}\Phi\right) = 0. \tag{5.7}$$

This will be referred to as the momentum equation.

Remark. If the absence of a source term for the vector field $\vec{m}$ seems unusual, it is of interest to note that, in the cited studies of Couette flow in superfluid Helium, the solid-body flow was always approximated by a constant. And that explains why it was never considered as an independent dynamical variable.

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4. Variation of the density:

$$\dot{\Phi} + \frac{1}{2}\dot{\vec{X}}^2 + \kappa\dot{\vec{X}} \cdot \vec{\nabla}\Phi - \frac{1}{2}(\vec{\nabla}\Phi)^2 - \frac{\partial f}{\partial \rho} - ST = 0. \tag{5.8}$$

These variational equations are conservation laws; together they imply the conservation of energy, $\mathcal{H}$ being the energy density. Note that two independent vector fields, $\dot{\vec{X}}$ and $-\vec{\nabla}\Phi$, are needed to get the correct signs for both squared velocity terms; opposite signs in (5.8), both positive in the Hamiltonian density. The term $\kappa\dot{\vec{X}} \cdot \vec{\nabla}\Phi$ does not contribute to the Hamiltonian, but it is crucial for the equation of continuity.

The stationary flow that was the main object of this study is a solution of the Euler-Lagrange equations. Numbers 2 and 3 are solved by taking ρ to depend on r only, $\dot{\Phi}$ constant, and

$$\Phi = t\dot{\Phi} + a\theta, \quad \vec{\nabla}\Phi = \frac{a}{r^2}(-y, x, 0), \quad \vec{X} = bt(-y, x, 0). \tag{5.9}$$

When these solutions are inserted into (5.8) we get, in the case of an ideal gas, for stationary flow,

$$\frac{a^2}{2r^2} - \frac{b^2}{2}r^2 + ab = C - (n+1)\mathcal{R}T. \tag{5.10}$$

This is the equation that tries to agree with the Navier-Stokes equation and it very nearly does, but to compare we must take the gradient.

$$\rho\vec{\nabla}\left(\frac{a^2}{2r^2} - \frac{b^2}{2}r^2 + ab\right) = -\vec{\nabla}p.$$

With the same flow, the Navier-Stokes equation is

$$\rho\vec{\nabla}\left(\frac{a^2}{2r^2} - \frac{b^2}{2}r^2 + 2ab\ln r\right) = -\vec{\nabla}p - \nu\vec{\nabla}\Delta\vec{v}.$$

Of course, we cannot reproduce the viscosity term in the Euler-Lagrange equations (but see below). The cross term ab is a constant, while in the Navier-Stokes equation it is replaced by the logarithm.

With the lagrangian density (5.1) a significant step in our program has been achieved. We now know that the idea of a viable lagrangian formulation of thermodynamics is not absolutely limited to potential flows. This leaves many things yet to be done, for example, the complete interpretation of the two vector fields that make up the total flow, the integration with electromagnetism, and with General Relativity. First of all we need to apply the new, tentative insight to a range of phenonomena; all that we know at present is that our new lagrangian density is suitable for cylindrical Couette flow in the laminar regime. A closely related phenomenon, the flow along plane, moving

walls or linear Couette flow will be examined below, in Section VII. Note that the new theory is new only in a context where both velocities are present; when one or the other is absent there is no deviation from the traditional approach.

The appearance, in the action principle for general flows, of an extra variable should not be surprising. The work of Feynman (1954), and of Landau and Lifshitz (1955, 1958) on the application of statistical mechanics to liquid Helium is based on a separate and different treatment of two types of excitations, the 'phonons' being accounted for by a gradient vector field and the 'rotons' understood as a different degree of freedom. These papers, and especially the much quoted paper by Hall and Vinen (1956) that they inspired, briefly refer to an action principle. What is minimized is a function $F' = F - I\Omega^2/2$, where the second term can be identified with the term $\rho \dot{\vec{X}}^2/2$ in our Lagrangian. (The term $-\rho(\vec{\bigtriangledown}\Phi)^2/2$ of our Lagrangian density is included in the free energy F.) In most of the papers that followed, the strong hint of a Lagrangian structure was systematically suppressed. See however, the more recent review by Fetter (2009).

Brenner's bi-velocity theory

This theory was invented to account for the fact that several different velocity fields are needed to describe certain fluids, especially fluids that carry electric charge. The flow of charge is not necessarily the same as the flow of mass (Brenner 2011, Brenner et al 2013). In that case the recognition of two densities leads naturally to two flow velocities. A concept of dual velocities is a feature of General Relativity as well, in Weinberg's treatment of conservation laws (Weinberg 1972). The theory developed in this paper also requires two separate velocities, but the interpretation is different and as yet not completely understood. We have seen that the velocity associated with momentum flow is different from the velocity of mass flow, but that observation is somewhat formal. For other references to bivector velocity fields see the quoted review by Martin (2010).

The viscosity

Most of the equations of motion represent conservation laws. In this respect we are in full agreement with the dominant approach to non equilibrium thermodynamics. Eq.(5.5) is the condition that the total mass be conserved. Eq. (5.6) can be interpreted as the conservation of 'momentum'. The Navier-Stokes equation relates the non conservation of momentum to the viscosity. To apply the theory to the case that the viscosity is not zero we add a source term to Eq.(5.7),

$$\frac{\partial}{\partial t}\left(\rho \dot{\vec{X}} + \kappa \rho \vec{\bigtriangledown}\Phi\right) = \mu \vec{\Delta} v, \quad \vec{v} := \kappa \dot{\vec{X}} - \vec{\bigtriangledown}\Phi. \tag{5.11}$$

To compare this with the Navier-Stokes equation we must combine it with the gradient of Eq. (5.8) (multiplied by ρ) to convert it to an equation for the time derivative of $\vec{v}$, this results in very close term-by-term agreement, including the viscosity term.

The addition of an extra term to the momentum equation is legitimate; it represents the influence of another degree of freedom that, because it is characterized by a longer time scale, can be approximated as a quasi-static development.

When the system of Euler-Lagrange equations is modified by the addition of the viscosity term in (5.11) the energy is no longer conserved, instead

$$\frac{d}{dt}\mathcal{H} = \dot{\vec{X}} \cdot \mu \Delta \vec{v}. \tag{5.12}$$

This not a complete description of the dissipative process; Eq.(5.12) was obtained under the condition that the entropy remains fixed, which is possible but not known to be true.

VI. Comparing two approaches
Predicting the flows.

A major success of the application of the Navier-Stokes equation to laminar, cylindrical Couette flow is the prediction, on the basis of the tangential component (4.2): $\mu \Delta \vec{v} = 0$, of just two kinds of flow, in this case potential flow and solid-body flow. Intuitively, the latter is a little unexpected; this flow is attained, or nearly attained, when the outer cylinder is driving and the inner cylinder is slipping. The Euler Lagrange equations associated with the lagrangian density (5.4) place no restrictions on the factor b in the solution $\vec{X} = b(-y, x, 0)$, but the inclusion of the viscosity as in Eq.(5.11) effectively implies that $\Delta \vec{v} = 0$ for stationary flows. The approach to viscosity is thus precisely the same in both theories.

The density profile.

The Euler Lagrange equation obtained by variation of the density, Eq,(5.8), reduces in the case that the flows are as in (5.9) to Eq. (5.10). This is in agreement with the Navier-Stokes equation except that the cross term ab is constant, while in the Navier-Stokes equation it depends logarithmically on the radius. If the angular velocity b were to vary with r, the term $\vec{X} \cdot \vec{\nabla}\Phi$ in the lagrangian density would no longer be a constant. If b were replaced by $br^{-\tau}$ we would obtain

$$\frac{1}{2}\dot{\vec{X}}^2 + \dot{\vec{X}} \cdot \vec{\nabla}\Phi - \frac{1}{2}(\vec{\nabla}\Phi)^2 = \frac{a^2}{2r^2} - \frac{b^2}{2}r^{2(1-\tau)} + abr^{-\tau} = C - (n+1)\mathcal{R}T. \tag{6.1}$$

This is in substantial agreement with Navier-Stokes in the limit of small τ. And it offers an extra free parameter for a small amount of fudging. In the case that the flow is either pure potential or pure solid-body there is full agreement.
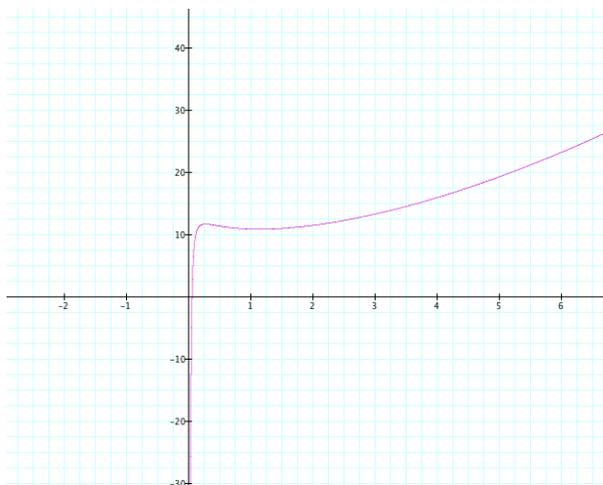
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Fig. 5. A temperature profile for cylindrical Couette flow. The small dip for small radius is produced by the *ab* term on the left hand side of (6.1). A similar feature has been observed.

The energy.

The principal reason for bringing out an alternative to the approach that relies heavily on the Navier-Stokes equation is of course the improved position of the energy concept. The equations of motion do not differ greatly from the equations used in the traditional method but, instead of an *ad hoc* formula for "energy" that is required to be conserved as an additional postulate (See for example Müller 2007) , we have a first integral of the equations of motion. We note that some authors require that the "energy equation" hold in consequence of the other conservation equations, thus expressing a point of view similar to ours. (Khalatnikov 1956.)

Instabilties

The main interest in Couette flow, from the very beginning, has been the question of instabilities of the laminar flow that is observed when the speed of rotation is increased. One approach to stability is perturbation theory. The unperturbed motion is of the form of 'solid-body' rotation, with fixed angular velocity. The considered perturbation affects only the potential field; this field is the only dynamical variable. This system is characterized by a Fetter-Walecka Lagrangian and it was treated as such in the review by Fetter already quoted. A complete treatment should include a variation of both vector fields. The successful application of the Navier-Stokes equation to the stability of flow in an incompressible fluids does not extend to thermodynamical systems defined by an equation of state. This may be related to the fact that the interaction between kinetics and heat is carried by the potential component of the flow. We propose that a more general approach to stability problems may give some new insight.

Let us emphasize this, that two velocity fields already appear in the traditional treatment of Couette flow, though one of them is frozen at the fixed, unperturbed value. Our suggestion is to treat both fields as independent field variables.

The expression $(n + 1)\mathcal{R}T$ on the right hand side is not valid at very low temperatures, but in the case of an ideal gas, with uniform entropy, it is proportional to a power

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of the density and it is perhaps reasonable to expect it to go to zero with the density. In that case one may expect the onset of turbulence to occur for a fixed value of the parameter

$$\frac{a^2}{2r^2} - \frac{b^2}{2}r^{(1-\beta)} + abr^\beta = C_0. \tag{6.2}$$

In the a, b plane this resembles a hyperbola open to the positive b axis. This too is in qualitative accord with experiments.

VII. Couette flows between plane walls

This example serves to show that one cannot expect the new action principle to agree with the Navier-Stokes equation in all cases, even in the limit of negligible viscosity. It is an example where the Navier-Stokes equation has very limited predictive power beyond the identification of the two principal modes of flow.

Here again is the Navier-Stokes equation,

$$\rho\left(\dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla})\vec{v}\right) = -\vec{\nabla}p - \mu\Delta\vec{v}. \tag{7.1}$$

Consider the problem of flow in the space bounded by plane walls parallel to the x, z-plane, $y = y_0$ and $y = y_1$. We limit our attention to stationary flows parallel to the x axis,

$$\vec{v} = (v, 0, 0),$$

with the function v depending only on y.

What the Navier-Stokes equation has to say is this. Projected on the direction of the flow, the velocity and the pressure are homogeneous (constant), and the equation reduces to

$$\mu\Delta v_x = 0.$$

with the general solution, when $\mu \neq 0$,

$$\vec{v} = (a + by)(1, 0, 0), \quad a, b \quad \text{constant.} \tag{7.2}$$

We are studying stationary flows, with $\dot{\vec{v}} = 0$; and since the velocity is constant along the direction of motion,

$$\frac{D\vec{v}}{Dt} := \dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla})\vec{v} = 0. \tag{7.3}$$

Consequently, all that remains is the equation

$$0 = -\vec{\nabla}p - \mu\Delta\vec{v}, \tag{7.4}$$

projected on the y-axis; the pressure gradient is balanced against the viscosity. Although the kinetic energy - if it is relevant to invoke energy in the context of the Navier-Stokes

equation - varies with the position, there is no force associated with this variation of the kinetic energy. It is true that the kinetic energy, in the special case that the flow is potential, is uniform, so that no force is associated with it, but the absence of a kinetic force when $b \neq 0$ tells us that this other type of flow is of a very different character, since a non zero gradient of the kinetic energy in any flow was expected to generate a force. Perhaps this is behind some hints found in the literature, to the effect that, in the context of the Navier-Stokes equation, the relation between linear flow and cylindrical flow is not perfectly understood (Faisst and Eckhardt 2000).

What is the meaning of (7.4)? One may, for example, assume that the pressure is that of an ideal gas,

$$p = \mathcal{R}\rho T,$$

but since the temperature is very rarely measured, perhaps the polytropic relation will be used instead,

$$p \propto \rho^{\gamma}.$$

Unfortunately the pressure also is rarely measured with the required precision. As far as the viscosity term is concerned the situation is less satisfactory, for usually, whenever measurements have been made, the purpose was to measure the viscosity. Consequently, it is very rare that an analysis is made in the interest of verifying a theory and all that can be said is that experimental results are not known to contradict the Navier-Stokes equation. Indeed, how could they?

It is unfortunate that few experiments on Couette flow are made with the idea of studying the laminar flow; the aim is usually to determine the conditions under which it breaks down to be replaced by more complicated flows including turbulence. Our purpose, so far, has been more modest, since we aim to understand laminar flow only, but more ambitious since we are looking for an application of a general theory with definite precepts including as far as possible a Lagrangian variational principle. The most urgent question is always the simplest: can we understand the observed laminar flow in the limiting case in which the role of viscosity is negligible?
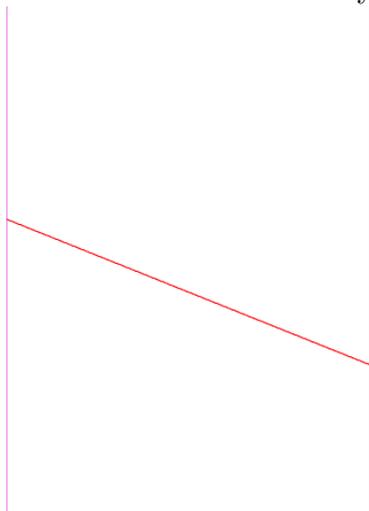
Fig. 6. The velocity profile of linear Couette flow according to the variational principle.

For simplicity, from here onwards, we set the parameter κ at unity. Our theory, in the present state of development, includes an equation of continuity that, when the flow has the form (7.2), illustrated in Fig. 5, namely

$$\vec{v} = \dot{\vec{X}} - \vec{\nabla}\Phi = (a + by)(1, 0, 0), \tag{7.5}$$

demands that the density depend on the coordinate y only. Solutions for the vector and scalar fields include the following

$$-\vec{\nabla}\Phi = a(1, 0, 0), \quad \vec{X} = bt(y, 0, 0). \tag{7.6}$$

The field equation is satisfied,

$$\partial_t(\rho\dot{\vec{X}} + \vec{\nabla}\Phi) = 0,$$

and finally there is the equation that comes from variation of the density; in the case of an ideal gas,

$$\frac{a^2}{2} - aby - \frac{b^2y^2}{2} = C - (n+1)\mathcal{R}T. \tag{7.7}$$

or

$$(n+1)\mathcal{R}T = \frac{1}{2}(by + a)^2 + \text{constant.}$$

This contrasts with the Navier-Stokes equation (7.4). The fluid is contained between moving walls located at two values of the coordinate y. If this interval includes the plane with coordinate $y = -a/b$, then the temperature has a minimum there, as in Fig. 7.
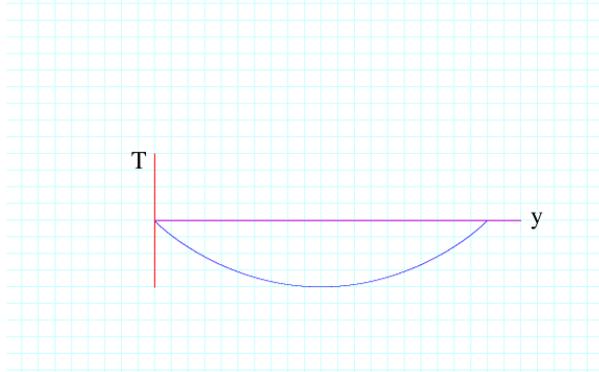

Fig. 7. The temperature profile of linear Couette flow between two walls

The order of magnitude of the variation of temperature is that of $v^2/\mathcal{R}$. If the velocity is 1 m/sec this will imply a temperature variation of order $10^4/C_p$, a very small variation indeed. So it is unlikely that this temperature variation has been observed.

It is nevertheless significant that, in remarkable contrast with the Navier-Stokes approach, our approach leads to a definite, quantitative prediction without the explicit intervention of viscosity, valid because the role of viscosity is merely to restrict the allowed velocity to the form (7.6).

Stationary flow past a flat or nearly flat surface continues to be a vital subject in ship design; see the recently evolving subject of stepped hulls.

VIII. Final remarks

• This paper was motivated by the need for an action principle formulation of General Relativity in the presence of matter. An action principle for hydrodynamics has been presented that, to encompass general velocity fields, relies on the introduction of two kinds of velocities. In hindsight there were many indications that this dual approach to fluid velocity was needed.

• The decision to make use of the Lagrangian velocity $\dot{\vec{X}}$ in hydrodynamics has startling consequences for relativistic physics. A precept that has dominated all relativistic fluid dynamics is that the relativistic 3-vector velocity field shall be upgraded to a relativistic 4-vector velocity field. The electric field is a 3-vector but is promoted to a tensor field. Attempts to imbed the field $\dot{\vec{X}}$ in a tensor field has led to the following field structure.

Consider a relativistic 2-form Y and set

$$\dot{\vec{X}}^i = \frac{1}{2}\epsilon^{ijk}Y_{jk}, \quad \eta_i = Y_{i0},$$

with latin indices running over 1,2,3. Then the Lorentz invariant scalar

$$dY^2 = \frac{1}{2}\left(\dot{\vec{X}} + \vec{\bigtriangledown}\wedge\vec{\eta}\right)^2 - \frac{c^2}{2}(\vec{\bigtriangledown}\cdot\vec{X})^2$$

is invariant under the gauge transformation $\delta Y = d\xi$. There is a unique gauge in which the field $\vec{\eta} = 0$ and this expression reduces to $\dot{\vec{X}}^2/2$. Other terms in the Lagrangian also can be made manifestly Lorentz invariant, while $\dot{\Phi} - \vec{\bigtriangledown}\Phi^2/2$ is the non relativistic limit of $(1/2c^2)g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}$, with $\psi = c^2 t + \Phi$ (Fronsdal 2007). In this form the theory becomes a relativistic Lagrangian field theory and the associated energy-momentum tensor is an appropriate source for Einstein's equation in the presence of a rotating source, a source that satisfies the Bianchi constraint.

• Relativistic electromagnetic theory of fluids was created by Minkowski using the 3-vector-to-4-vector prescription. If we are going to use the velocity field $\dot{\vec{X}}$ then the theory of relativistic electromagnetism of fluids will have to be recreated to accommodate.

• Finally it is clear that the proposed development of the structure of velocity space may have profound effect on hydrodynamic stability studies, especially where it is needed most: in the case of incompressible fluids where the subject is thermodynamics rather than just hydrodynamics.

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