

## VIII. SUPERFLUID HELIUM

Revised 3/19/13.

### VIII.1. Experiments and interpretation

Here is a very brief summary of the first important experiments on superfluid Helium, the development that culminated with Landau's two fluid theory. It is natural that we should take a special interest in this system, for it stands alone as the only important success of two fluid thermodynamics. The matter of (quantized) vortices will not be taken up at this point.

#### A. The fountain experiment

Liquid Helium is cooled in vessel A, Fig. 8.1.1. A capillary that connects this vessel to the vessel B is very fine and no significant amount of liquid gets through, until the temperature descends below 2.1K. Both vessels are then thermally isolated. It is observed that, as liquid flows from A to B through the capillary, vessel A heats up and vessel B cools down: heat flow is in the direction opposite to the mass flow.\*

This unusual behavior led F. London (1938) to suggest that liquid Helium below 2.19 degrees Kelvin is a mixture of two components or phases, a normal component and a superfluid component with vanishing viscosity and entropy, and that the liquid that gets through (the superfluid component) carries no entropy with it. The capillary, with its capacity for denying passage to the normal component, plays the part that is taken by gravitation in the case of saturation. That the entropy of the super fluid component is near zero (supposing that it is) is a reflection of the spectrum, the information can be used but not justified within our context. It suggests that the free energy of this component is independent of the temperature, which makes sense when the total free energy is regarded as a sum  $f_1 + f_2$  of those of the two separate gases. A reasonable hypothesis is that the temperature of the liquid in equilibrium cannot be lowered except by conversion of some of the liquid from normal fluid to superfluid.

This behavior does not imply that liquid helium is governed by a different kind of thermodynamics; we shall see, in Section 8.2.3, that London's intuition about the entropy is sufficient to explain the unusual behavior of this liquid.

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\* This would appear to be in direct contradiction of one of the statement of the second law offered by Clausius: "Heat will not flow from cold to hot if nothing else is going on." It is not the clearest statements of the second law of thermodynamics. A deeper study of the fountain effect, including the role of gravitation, will be undertaken in a later chapter.

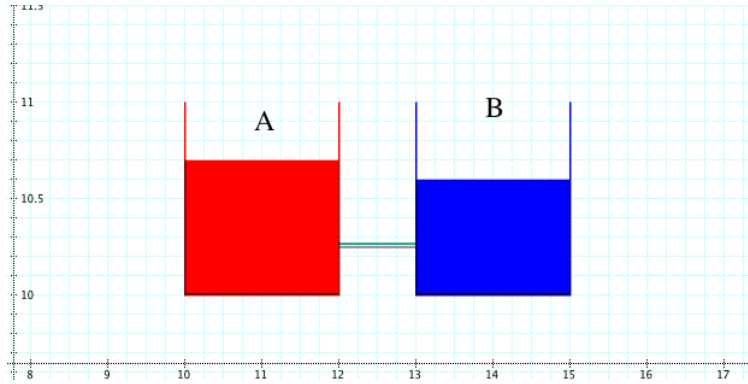


Fig.8.1.1. The fountain effect. The central portion (light blue) is filled with a fine powder that is impermeable to the normal form of liquid helium.

### B. The entropy

We have proposed the representation

$$s = \rho_1 S_1 + \rho_2 S_2, \quad (8.1.1)$$

of the entropy density of a mixture. The specific entropies  $S_1$  and  $S_2$  are adiabatic invariants. In the analogous case of saturation, heating results in an increase of the amount of gas while the specific entropies  $S_1$  and  $S_2$  remain fixed or almost so. The fountain effect can be similarly understood as resulting from the fact that  $S_1$  and  $S_2$  in He II vary but slowly with the temperature and that  $S_2 \ll S_1$ .

Very precise measurements of the entropy density are available. In the upper half of the temperature interval, where the normal gas dominates, the first term is expected to make the larger contribution and on this assumption it is seen that the specific entropy  $S_1$  varies only slowly with the temperature, which confirms the view that the lowering of entropy takes place, mainly, through conversion of the normal fluid to superfluid.

**TABLE 8.1.1**

$p = 00.0,$	$s = 1.59\rho_1 + .0058 T^3 \rho_2,$
$p = 05.0,$	$s = 1.450\rho_1 + .0042 T^3 \rho_2,$
$p = 10,0$	$s = 1.36\rho_1 + .0025 T^3 \rho_2,$
$p = 15.0,$	$s = 1.289\rho_1 + .0025 T^3 \rho_2,$
$p = 20.0,$	$s = 1.18\rho_1 + .0020 T^3 \rho_2,$
$p = 25.0,$	$s = 1.129\rho_1 + .0017 T^3 \rho_2.$

Table 8.1.1. Relation between measured entropy density and the equilibrium values of densities and temperature, covering the whole interval from .1K to 2.1K. Note that entropies given by MacDonnelly are ‘specific densities’. Our “s” is density in  $J/cm^3$ ; the entropy quoted by WacDonnelly multiplied by the total density  $\rho$ . (The numbers for  $p = 0,10, 20$  have been recalculated and corrected by very small amounts.)

There is no significant dependence of the entropy on the temperature when  $1 < T < 2$ . Towards lower temperatures, when the normal gas is absent, the entropy of the super fluid becomes dominant and there is no way to go except to reduce the coefficient of  $\rho_2$ .

### C. The equilibrium configuration

A relation between the basic variables  $\rho_1, \rho_2$  and  $T$  defines the equilibrium configurations of the system. Using data found in Putterman's book we plot all available physical points in the interval  $1.3 < T < 2.1$  in Fig.8.1.2 and find that the following equation describes it well,

$$X(\rho_1, \rho_2, T) := (\rho_1 + a\rho_2 + b)(\rho_1 + c\rho_2 + d) + 10^{-4} = 0,$$

with

$$a = -.15, \quad b = .014 - .21(T - 1.3)^2, \quad c = 1.86 - .31T, \quad d = -.398 + .113T.$$

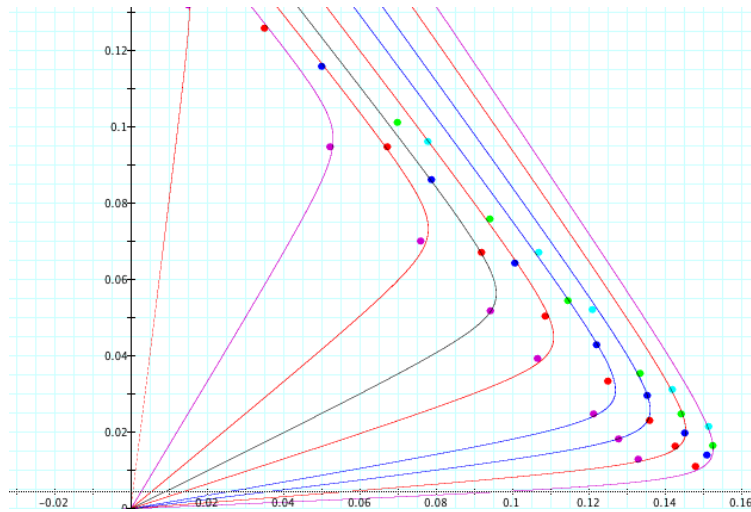


Fig.8.1.2. The set of equilibrium configurations as given by Putterman. The color goes with the pressure;  $p = 0$  for the left most purple dots and  $p = 25$  for the (also purple) dots at the right end. The lines connect points with the same temperature, 1.3 K for the outer line and 2K for the inner line. The extrapolation from the purple dots to the origin (negative pressure) is pure fantasy.

Using data collected by Brooks and McConnellly we get a slightly different result, see Fig.8.1.3. A more convenient expression for the equilibrium relation between  $\rho_1, \rho_2$  and  $T$  will be found later.

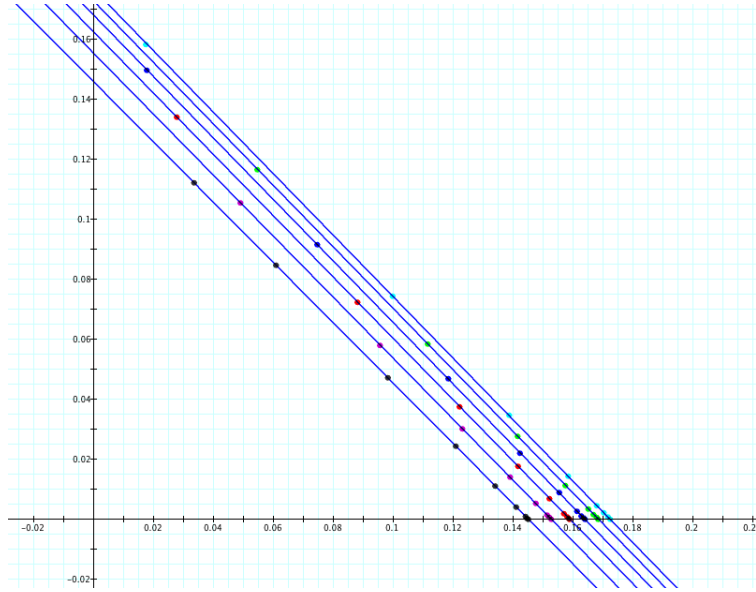


Fig.8.1.3. The set of equilibrium configurations according to Brooks and Donnelly. Blue lines are loci of equal pressure.

The data points used are tabulated at the end of this chapter.

Values of the two densities can be represented with high precision as follows, for each of six values of the pressure, the densities in  $mg/cm^3$  and the pressure in atmospheres. The variation with pressure, though relatively weak, makes this representation inconvenient.

**TABLE 8.1.2**

$$\begin{aligned}
 p = 00, & \quad \rho_1 = 1.90 T^{5.5}, & \rho_2 = 145.0 - 1.74 T^{5.6}, \\
 p = 05, & \quad \rho_1 = 2.17 T^{5.6}, & \rho_2 = 152.5 - 1.86 T^{5.8}, \\
 p = 10, & \quad \rho_1 = 2.56 T^{5.7}, & \rho_2 = 159.0 - 2.20 T^{5.9}, \\
 p = 15, & \quad \rho_1 = 3.03 T^{5.8}, & \rho_2 = 164.0 - 2.65 T^{6.0}, \\
 p = 20, & \quad \rho_1 = 3.70 T^{5.9}, & \rho_2 = 168.5 - 3.20 T^{6.1}, \\
 p = 25, & \quad \rho_1 = 4.40 T^{6.0}, & \rho_2 = 172.5 - 4.00 T^{6.2},
 \end{aligned}$$

Table 8.1.2. This representation of the concentration is valid over the whole temperature interval, from .1K to 2.1K.

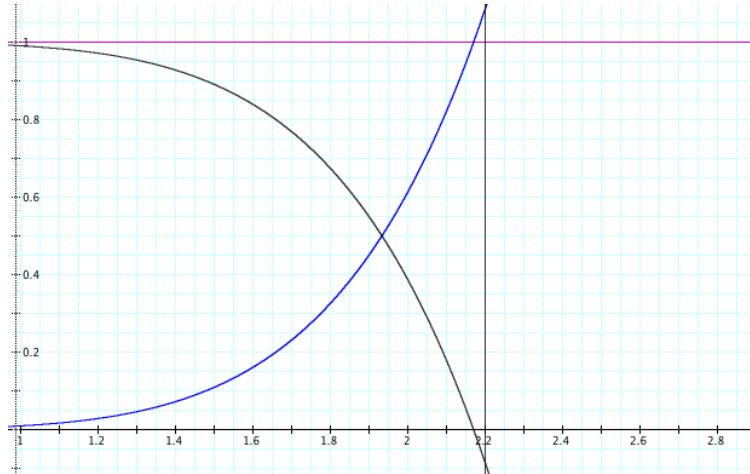


Fig.8.1.4. Sketch of the concentrations of the normal fluid (increasing) and the superfluid component (decreasing) between  $T = 1K$  and  $T = \lambda$ , for fixed pressure.

#### D. Second sound

Landau's theory is distinguished by the fact that it treats the two components of a mixture as independent degrees of freedom. It predicted the existence of two kinds of sound, a normal mode (first sound) in which both oscillations are in phase and another, second sound, in which they are in opposite phase. This was a real prediction, made before any experiments confirmed the existence of two sounds in liquid Helium.

With the advantage of already knowing the existence of two sound modes we can affirm with some assurance that there must be two densities and two velocities. With the limitation to irrotational vector fields the kinetic part of the lagrangian density is

$$\rho_1(\dot{\Phi}_1 - \vec{v}_1^2/2 - \phi) + \rho_2(\dot{\Phi}_2 - \vec{v}_2^2/2 - \phi). \quad (8.1.2)$$

The appearance of an additional degree of freedom naturally implies that two sound modes are likely to exist; the surprise is not the existence of two sounds in liquid Helium but the fact that a similar phenomenon is rare in other mixtures. The reason is, as was seen, that most mixtures are characterized by a strong interaction between the components, interactions that effectively quash the mode in which the two fluctuations are in opposite phase.

Unless strong interactions between the two components prevent it, two sounds come naturally. \*

An implication of this is that we shall have two equations of continuity,

$$\dot{\rho}_i + \text{div}(\rho_i \vec{v}_i) = 0, \quad i = 1, 2. \quad (8.1.3)$$

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\* We have included the gravitational potential  $\phi$  in order to make a point. According to General Relativity, this potential is a natural companion to the operator  $d/dt$ . We are therefore very reluctant to include a time derivative anywhere else, which makes us skeptical of Landau's theory.

The observation that only the total mass is conserved suggests that there can be only one equation of continuity; this is a puzzle that is solved in Landau's theory by giving a dynamical role to entropy (while the dynamical role of  $\rho_1/\rho_2$  is obscure). But we remember that conservation of mass is not an automatic consequence of the equation of continuity; it only follows only when, in addition, one imposes the condition that the boundary velocity be tangential to the boundary. We follow Landau in that we shall impose this boundary condition on the total flow

$$\vec{j} = \sum_i \rho_i \vec{v}_i,$$

but not on the individual velocities. Both mass densities are conserved in the bulk but only the total mass is preserved in the interaction with the boundary. The variations of the densities under which the action is extremal are constrained by the fixing the total mass, but not the individual masses.

### E. Thermodynamic pressure and the pressure on the walls

The thermodynamic pressure is defined as

$$p = \rho_1 \frac{\partial f}{\partial \rho_1} + \rho_2 \frac{\partial f}{\partial \rho_2} - f. \quad (8.1.4)$$

The free energy is a function of the three independent variables  $\rho_1, \rho_2$  and  $T$ . It is common usage in thermodynamics of mixtures since Gibbs, and especially so in Landau's treatments of liquid Helium, to reduce the concentration to the role of a parameter, denying it the respect that it deserves as a thermodynamic variable. Such treatments are dynamically incomplete, being partially on shell.

The pressure on the wall has an additional contribution whenever the wall is maintained at a temperature that is different from the fluid. In the case of a heated wall, heat is transmitted to the liquid by the conversion of superfluid to normal fluid. This upsets the balance and, to compensate, normal fluid flows away from the wall and super fluid flows toward the wall. This inelastic reflection creates a reactive force on the wall, which has been measured (Hall, 1954). The standard interpretation of this effect gives results that are in qualitative agreement with experiment. We have nothing to add to it, except this: The ambient thermodynamic pressure remains the same; there is a flow, but the thermodynamic pressure in the bulk is given by (8.1.4).\*

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\* Some aspects of Landau's theory reflect the prejudice, common since the original work of Gibbs on heterogeneous fluids, of regarding the mixture as a unary thermodynamic system with the concentrations reduced to parameters. In this view it is tempting to regard the "internal" kinetic energy as a part of the free energy. In the case of superfluid helium it has led to the suggestion that the thermodynamic pressure contain a term proportional to  $(\vec{v}_1 - \vec{v}_2)^2$ . (Putterman 19?? ) I hope to be able to maintain that the thermodynamic pressure is given by Eq.(8.1.4), as always.

Fig.8.1.5. Hall's experiment.

The on shell pressure can be expressed in terms of  $T$ ,  $\rho_1$  and  $\rho_2$ , this relation is found by measurements at equilibrium to take the following form; for example,

$$T = .1 : \quad p = 25 - 1360(.17246 - \rho) + 14300(.17246 - \rho)^2 \quad (8.1.5)$$

The derivative,

$$\frac{dp}{d\rho} = 28600\rho - 3.6384 = c^2$$

varies from 526.7 at  $p = 0$  to 1308.8 at  $p = 25$ . The unit of pressure is one atmosphere. To convert to cgs units multiply by  $10^6$ , to obtain the square of isothermal velocity of normal sound in cm/sec. At  $T = .1$ ,  $c = 229m/sec$  at  $p = 0$  and  $c = 362m/sec$  at  $p = 25$ . At higher temperatures much the same. The quasi independence of the speed of the temperature is significant.

### F. Critical behavior

According to Arp (1989), all singular features of the thermodynamic functions can be fitted with a specific heat that is singular at  $T = \lambda$ , where  $\lambda$ , a function of the density, is the  $\lambda$ -temperature expressed in terms of  $\rho$ . A term like

$$(T - \lambda) \ln\left(1 - \frac{T}{\lambda}\right)$$

in the free energy will produce just such a term in the heat capacity.

Although we shall use a similar phenomenological function, we are not going to take the point of view of Arp, who emphasized this singular behavior in a trial function with 41 adjustable coefficients. Arp's very thorough work suffers from an important omission: it deals with a single fluid. There is no account of the separate densities of the two components, consequently no mention of second sound. All thermodynamic relations ignore the concentration.

Let us remember that van der Waal's highly successful modeling of a gas near and above the critical gas/liquid point did not predict the "correct critical behavior". This refers to

a very small region that involves altogether different concepts. The finer details of critical behavior is beyond the scope of thermodynamics and such behavior cannot serve as a principal guide to the analytic behavior of thermodynamic functions in the larger context.

## VIII.2. First analytic models

We shall construct idealized lagrangian models, with features that resemble those of He II, but simplified and, in one respect, exaggerated.

**A. Thermodynamic variables.** We postulate that the correct choice of thermodynamic variables are  $T$ , the densities  $\rho_1, \rho_2$  and the individual, specific entropies  $S_1, S_2$ . We find that it is the most convenient choice though others are of course possible. In what follows we are concerned almost exclusively with equilibrium configurations; with no flow ( $\vec{v}_1 = \vec{v}_2 = 0$ ) and all fields uniform.

**B. Structure of the lagrangian.** It is expected that the lagrangian density, in a first approximation, is a sum of three kinds of contributions, the kinetic terms (8.1.2), the free energy, and the entropy in the form (8.1.1). Principal terms in the free energy are associated with each component separately,

$$f = f_1(T, \rho_1) + f_2(T, \rho_2) + \dots ,$$

but additional terms will be needed to take interactions into account.

An adiabatic system is defined by fixing the specific entropies  $S_1, S_2$  and the pressure (or the volume). The way that the total entropy is apportioned between the two species can be understood in terms of an underlying atomic theory; for He II it can be determined experimentally and for our model it must be freely invented. In the event that a successful model is found it must be reconciled with atomic theory.

**C. The pressure.** The term attributed to  $f$  so far, Eq.(8.1.2), being linear in the density, produces no thermodynamic pressure. In He II a good approximation is (8.1.5), with variations of  $c^2$  of up to 5 percent over the range of temperatures. Our models all have pressure

$$p = c^2(\rho - .1455), \tag{8.2.1}$$

with  $c^2$  constant. This pressure can be derived from from another term to be included in the free energy, for example

$$c^2(\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2 + .1455), \tag{8.2.2}$$

The absence of any dependence on the temperature in the expression (8.2.1) for the pressure ensures that the velocity of propagation of ordinary (first) sound in the models is constant. In He II it varies from 227 to 366 m/sec.



**D. Entropy.** The distribution of the entropy between the two species, as a function of pressure and temperature, is not determined within adiabatic thermodynamics. Experimentally, for liquid helium, what is known is summarized in Table 8.1.1. Any model must make some assumption, to be justified either by experiment or by atomic theory, preferably by both. Our experience with phase transitions, dissociation and immiscible fluids is not very helpful in the present instance. The only reliable guide is observation.

We take as a principal feature of our idealized model the axiom that the measured entropy is carried by the normal component; that is,

$$s = \rho_1 S_1. \quad (8.2.3)$$

This defines a family of adiabatic systems, parameterized by a single parameter  $S_1$ . The value of  $S_1$  determines the adiabatic system and its equilibrium configuration, just as is the case for unary systems. This feature emulates one of the most dramatic aspects of He II, in an extreme form, by setting  $S_2 = 0$  while for He II it is insignificant except at the lowest temperatures.

In this case it is natural to suppose that the temperature dependent terms in the free energy do not contain the density  $\rho_2$ . We make this another postulated property of our model. The simplest possibility is that the  $T$  dependence of  $f$  is confined to a term

$$a(T)\rho_1. \quad (8.2.4)$$

Linearity in  $\rho_1$  implies that this term makes no contribution to the pressure. The adiabatic condition is

$$S_1 = -\partial a / \partial T.$$

If  $a$  were linear in  $T$ , then the adiabatic condition would fix the value of  $S_1$  once and for all; it would also make the internal energy independent of the temperature. Yet it is a feature of all familiar systems that the internal energy is first order in  $T$ . To achieve this,  $a(T)$  must include terms of the type  $T \ln T$ . Since a singularity is expected at  $T = \lambda$  we are tempted to follow Arp, including a term  $b(\lambda - T) \ln(\lambda - T)$ , with  $b$  and  $\lambda$  constant. We leave the function  $a(T)$  undetermined for the present.

We note that the model adiabatic system has a fixed temperature. In this it resembles a saturated system where heat is added at constant temperature and pressure. The entropy density  $s$  is not fixed, since it increases with the partial density  $\rho_1$ . The specific entropy  $S_1$  varies slowly. Before we can calculate the heat capacity we must complete the construction of the lagrangian.

**E. Sound speeds.** So far our partial expression for  $f$  contains no coupling between the two nodes. This implies two sound modes, one in which the super fluid remains at rest, another where the normal component is quiet. This is, of course, very different from what is observed. So we need to include an interaction term in the free energy.

Two kinds of coupling are possible: through direct, non thermal interaction between the two components, and via the intermediary of the temperature. If we are going to maintain the fundamental property of the model, that the super fluid carries no entropy, then the

only option that is available is to add an interaction term that involves both densities but not the temperature. Guided by our previous experience with sound propagation in mixtures we would add a term  $\alpha\sqrt{\rho_1\rho_2}$  to the free energy density, but let us make the more general assumption that the interaction takes the form

$$f_{\text{int}} = \alpha(\rho_1, \rho_2).$$

For a small deviation from an equilibrium mode we have the following equations

$$\frac{\ddot{\rho}_i}{\rho_i} = -\dot{v}_i = \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial \rho_i} (f + sT),$$

where  $f$  is the sum of the three terms (8.1.2), (8.2.2) and (8.2.4). Ignoring constants,

$$\frac{\partial}{\partial \rho_1} (f + sT) = a(T) + c^2 \ln \rho_1 + S_1 T + \alpha_1, \quad (8.2.5)$$

$$\frac{\partial}{\partial \rho_2} (f + sT) = c^2 \ln \rho_2 + \alpha_2, \quad (8.2.6)$$

where suffixes on  $\alpha$  stand for derivatives. The adiabatic condition makes  $T$  constant and the  $T$  dependent terms do not contribute. Hence

$$\begin{aligned} \frac{\ddot{\rho}_1}{\rho_1} &= c \frac{\rho_1''}{\rho_1} + \rho_1 \alpha_{11} \frac{\rho_1''}{\rho_1} + \rho_2 \alpha_{12} \frac{\rho_2''}{\rho_2}, \\ \frac{\ddot{\rho}_2}{\rho_2} &= c \frac{\rho_2''}{\rho_2} + \rho_2 \alpha_{22} \frac{\rho_2''}{\rho_2} + \rho_1 \alpha_{21} \frac{\rho_1''}{\rho_1}. \end{aligned}$$

The squared-speed matrix is

$$c^2 + \begin{pmatrix} \rho_1 \alpha_{11} & \rho_2 \alpha_{12} \\ \rho_1 \alpha_{21} & \rho_2 \alpha_{22} \end{pmatrix}.$$

The determinant of the matrix (without the  $c^2$  term) is

$$\rho_1 \rho_2 \det(\alpha_{ij}).$$

The case that the function  $\alpha$  is everywhere singular; that is when the determinant vanishes identically, is especially easy to analyze, for in that case the eigenvalues are

$$c_1^2 = c^2, \quad c_2^2 = c^2 + \rho_1 \alpha_{11} + \rho_2 \alpha_{22}.$$

If  $\alpha$  is zero either component moves independently, with the same speed since  $\partial p / \partial \rho_1 = \partial p / \partial \rho_2$ . If instead we had replaced the pressure potential  $c^2(\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2)$  by the expression favored by Arp and others, namely  $c^2 \rho \ln \rho$ , then only the total density would oscillate; that is, one of the speeds would be zero. It so happens that this is much closer to the facts in the case, so that the expression

$$c^2 \rho \ln \rho = c^2(\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2) + c^2(\rho_1 \ln \frac{\rho}{\rho_1} + \rho_2 \ln \frac{\rho}{\rho_2})$$

is close to being the correct expression to use in  $f$ . To obtain a non zero speed for second sound we need only reduce the strength of the second, interaction term. Thus

$$f_{\text{int}} = \alpha(\rho_1 \ln \frac{\rho}{\rho_1} + \rho_2 \ln \frac{\rho}{\rho_2}), \quad (8.2.7)$$

with the constant  $\alpha$  very close to  $c^2$ .

We note that two terms in  $f$  combine according to

$$c^2(\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2) + f_{\text{int}} = \alpha \rho \ln \rho + (c^2 - \alpha)(\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2).$$

Only the second term contributes to the equilibrium condition.

Now the speed-squared matrix becomes

$$c^2 + \frac{\alpha}{\rho} \begin{pmatrix} -\rho_2 & \rho_2 \\ \rho_1 & -\rho_1 \end{pmatrix},$$

with eigenvalues  $c^2$  and  $c^2 - \alpha$ . With a slight change of notation, the complete expression for  $f$  takes the form

$$f = A + B, \quad A = a(T)\rho_1, \quad (8.2.8)$$

$$B := c_1^2 \rho \ln \rho - c_2^2(\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2) + C_0 + C_1(\rho_1 - \rho_2),$$

where  $c_1(c_2)$  is the speed of first (second) sound,  $c_1 = c$ , and  $C_0, C_1$  are constants.

The values of the parameters  $c_1, c_2$  can be chosen to agree with the two speeds of He II.

**F. Equilibrium.** This is a state of rest, with both velocities equal to zero. It is determined by equalizing the chemical potentials,

$$\left. \frac{\partial}{\partial \rho_1} (f + sT) \right|_{\rho, T} = 0. \quad (8.2.9)$$

The relation that determines the concentration at equilibrium is thus

$$\tilde{a}(T) + c_2^2 \ln \frac{\rho_2}{\rho_1} + 2C_1 = 0, \quad (8.2.10)$$

where the function  $\tilde{a}$  is

$$\tilde{a}(T) = a - T(da/dT).$$

The model is characterized by a fixed relationship between concentration and temperature, independent of  $\rho$  and  $p$ . In liquid Helium below the  $\lambda$  point the concentration is often quoted as a fixed function of the temperature. In the model this relationship between concentration and temperature is determined by the choice that we make of the function  $a(T)$ , and we may choose this function so as to further enhance the similarity of the model to He II.

### On the heat capacities of mixtures

The free energy is a function of the temperature and the densities. The internal energy is a function of the entropy and the densities, defined as follows. A density  $\tilde{u}$  is defined by

$$\tilde{u}(\rho_1, \rho_2, T) = \left( f - T \left. \frac{\partial f}{\partial T} \right|_{\rho_1, \rho_2} \right)$$

and the internal energy density  $u(\rho_1, \rho_2, s)$  is defined by

$$u(\rho_1, \rho_2, s) = \tilde{u}(\rho_1, \rho_2, T) \Big|_{T=\tilde{T}(\rho_1, \rho_2, s)},$$

where the temperature is evaluated at the point (assumed to be well defined) obtained from the adiabatic condition. This function is not what concerns us here; it is the function  $\tilde{u}$  that is of more immediate interest.

In the case that there is only one component, the heat capacity  $C_\rho$  or  $C_V$  is defined by

$$C_\rho = \frac{\partial \tilde{u}}{\partial T} \Big|_\rho.$$

It is measured by adding heat to the system while keeping the density fixed.

In the case of a heterogeneous system without chemical reactions or phase changes, what is measurable and usually quoted as  $C_V$  is

$$C_{\rho_1, \rho_2} = C_{\rho, x} = \frac{\partial \tilde{u}}{\partial T} \Big|_{\rho_1, \rho_2}.$$

It is measurable and well defined because the configuration is fixed for each value of  $T$  by the values chosen for  $\rho_1$  and  $\rho_2$  or  $\rho$  and  $x = \rho_1/\rho_2$ .

But in the case of He II and other systems with two mixed phases it is not usual, and perhaps not possible, to keep the concentration fixed. Heating takes the system through a sequence of equilibrium states in which the concentration is determined by  $\rho$  and  $T$ . We have

$$d\tilde{u} = \frac{\partial \tilde{u}}{\partial T} \Big|_{\rho_1, \rho} dT + \frac{\partial \tilde{u}}{\partial \rho_1} \Big|_{\rho, T} d\rho_1 + \frac{\partial \tilde{u}}{\partial \rho} \Big|_{\rho_1, T} d\rho.$$

Since the second term is zero at equilibrium we have

$$C_V = C_{\rho, T} = \frac{\partial \tilde{u}}{\partial T} \Big|_{\rho_1, \rho_2} = \rho_1 \frac{d\tilde{a}}{dT}. \quad (8.2.11)$$

I expect that what is done for He II is to fix the volume and pay no attention to the pressure, or vice versa; the difference is in any case practically negligible.

**G. Heat capacity.** The extreme form of the model entropy, although it reminds us of He II, actually prevents a close approximation of the density profiles, so we present instead a simple choice that allows all calculations to be done analytically. We favor the upper range of temperatures,  $.6 < T < \lambda$ .

The data that we have used does not include reflect the singular behavior at  $T = 0$  or at  $T = -\lambda$ . Instead, a plot of  $C_V/\rho_1$  reveals a cusp just below  $T = 1$ . The following analytic expression for  $a$ ,

$$a(T) = a(T - b) \ln |T - b| + cT \ln(T), \quad \tilde{a}(T) = -ab \ln |T - b| - (a + c)T. \quad (8.2.11)$$

gives close agreement with the observed properties of Helium II. The constraint (8.2.10) gives a fixed relation between  $T$  and the concentration, illustrated in Fig.8.3.1-3. The heat capacity is shown in Fig.8.3.4. The entropy is

$$\frac{s}{\rho_1} = -\frac{da}{dT} = -a - a \ln |T - b| + c + \ln T, \quad (8.2.12)$$

and the heat capacity,

$$\frac{C_V}{\rho_1} = \frac{d\tilde{a}}{dT} = \frac{-ab}{|T - b|} + a + c. \quad (8.2.14)$$

The equilibrium condition (8.2.10) gives another relation among experimental quantities. Because of the extreme assumption that defines the model,  $S_2 = 0$ , we expect it to be quite different from He II at the lowest temperatures.

### VIII.3. Model vs He II

The condition of equilibrium can be examined by plotting both sides of Eq. (8.2.10), adjusting the parameters to make a best fit. Several good fits can be found; in what follows we have restricted attention to the measurements at  $p = 0$ . The simplest is to take  $b = 0$ ,

$$\ln \frac{\rho_2}{\rho_1} = 10 - 5.25 T. \quad (8.3.1)$$

This is a very good fit for  $T$  in the upper interval  $.8 < T < \lambda$ . Since, for He II,  $c_2^2$  is close to unity, we may take

$$a = c = 0, \quad b = -6 \quad (8.3.2)$$

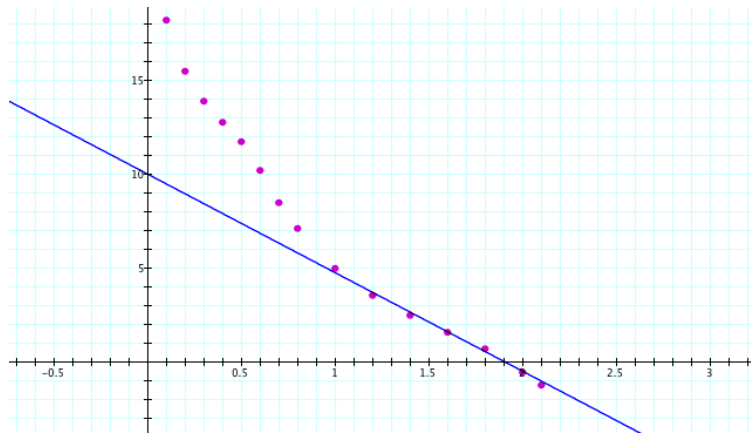


Fig.8.3.1. First fit to concentration vs. temperature with Eq.(8.3.1).

A very good fit for the entire range of temperatures is obtained with

$$\ln \frac{T}{\lambda - T} = 6 \ln \frac{\rho_1}{\rho_2}, \quad (8.3.3)$$

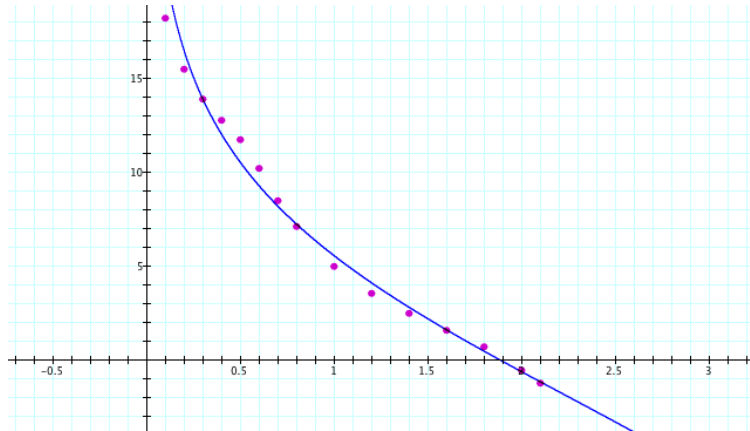


Fig.8.3.2. Second fit to concentration vs. temperature, with Eq.(8.3.3).

But a good fit to the equilibrium condition (8.2.10) that at the same time gives a good fit to the specific heat, Eq.(8.2.11) is

$$\ln \frac{\rho_2}{\rho_1} = -1.5 \ln |T - .64| - 4T + 7.6. \quad (8.3.4)$$

As seen in Fig.8.3.3 and Fig.8.3.4 it reproduces the the behavior of Helium in the upper interval of temperatures,  $.6 < T < 2.1$ . At the same time it requires that

$$\frac{C_V}{12\rho_1} = \frac{1.5}{|T - .64|} + 4.$$

This is compared with experiments in Fig 8.3.5)

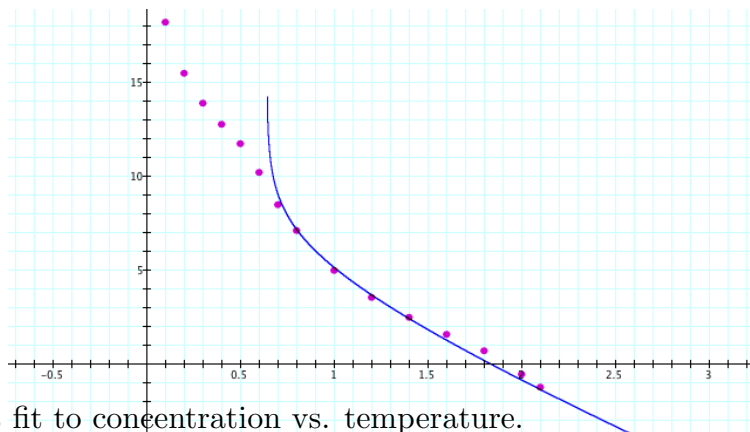


Fig.8.3.3. Best fit to concentration vs. temperature.

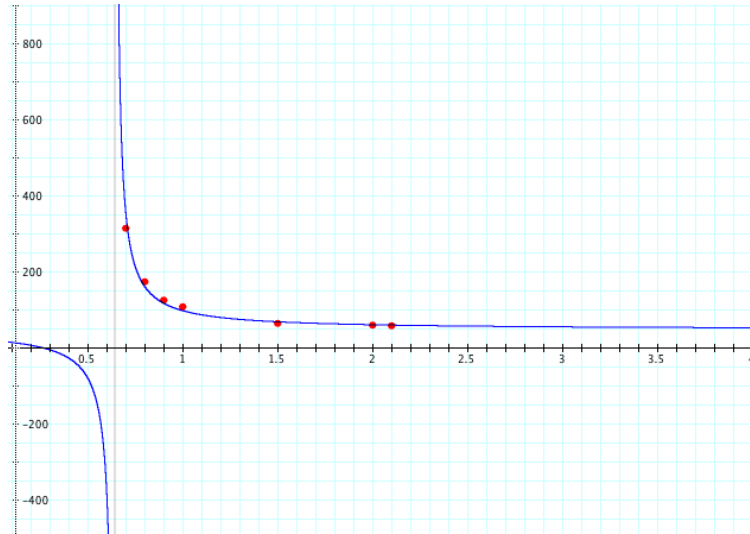


Fig.8.3.4.  $V/\rho_1$  vs. temperature, Eq.(8.3.4.new). See SuperCV.Jan.1.gcf Continue to left

A better model of He II must take into account that  $S_2$  is not quite zero; in fact, it is quite significant at the lowest temperatures.

All of the functions vary over many orders of magnitude, which makes it difficult to bring out detailed structure. But the function  $C_V/s$  varies little, and the plot, Fig.8.3.3, reveals a cusp at  $T = 1$  or slightly lower, close to the bend in Fig.8.3.1.

### The physical interpretation; temperature

We have shown that some of the stationary properties of the model show a marked affinity to super fluid Helium. What is more remarkable is that some dynamical features are also represented, in spite of a markedly different physical interpretation.

The model incorporates London's insight concerning entropy, and it agrees with Landau's approach to two-fluid dynamics, up to a point. What is not clear in Landau's theory is the thermodynamic interpretation. A set of equations is produced to account for the phenomena, but the dynamics that they describe appears to be non adiabatic. To explain that statement consider the propagation of sound, in normal thermodynamics and according to the model.

The Gibbsean action principle regards the pressure and the entropy as fixed parameters. It is not usual to lump both variables together in this way, but we have found it instructive

to do so. To generate a sound wave in an ideal gas we interfere with the adiabatic system by driving a periodic variation of the pressure locally at one of the walls. A periodic variation of the density is generated, the thermodynamic pressure in the bulk is determined by the equations of motion. The motion is adiabatic away from the wall, in the sense that the Euler Lagrange equations are satisfied. Included in this system of equations is the adiabatic condition that relates the temperature to the density, and the specific entropy is fixed.

To generate normal sound in the super fluid the same procedure can be used. Second sound can also be generated mechanically, by vibrating a membrane that is permeable to the superfluid. But historically what is more significant is that second sound can be generated by temperature fluctuations at the wall. In Landau's theory the temperature is one of the dynamical variables that appear in the equations of motion, so a traveling wave is quite naturally excited. In the model the interpretation is more precise. Adiabatic dynamics has  $S_1$  and  $S_2$  fixed, the latter equal to zero. The adiabatic condition fixes the temperature in terms of  $S_1$ . The presence of a heat source at a wall perturbs the adiabatic system by providing an oscillating input and withdrawal of heat, changing the entropy. More significantly, this produces a fluctuation of the densities. As this density waves propagates adiabatically it obeys the adiabatic equations of motion with fixed entropy parameters, and with fixed temperature. It is worth pointing out that the relation (8.2.10) between concentration and temperature is not adiabatic; it holds only at equilibrium. It is not inconsistent to hold that a density wave can exist at a fixed temperature.

### **The physical interpretation; concentration**

In Landau's theory the concentration seems to be reduced to a parameter; more precisely to a fixed function of the density and the temperature. This function is often referred to as an equation of state; nothing in the dynamics gives an indication of what it is. This situation is analogous to hydrodynamics, derived from thermodynamics by eliminating the temperature using the adiabatic condition. The reduced theory is capable of describing some of the dynamics, but it is lacking in that it gives only a partial view of the more complete, underlying thermodynamics. Since, in Landau's theory, the concentration is not an independent adiabatic variable the propagation of dual sounds relies on oscillations of the temperature.

To compare our model with the phenomenological approach of Arp and others we can use the equilibrium condition to eliminate the concentration, to produce a partially on shell theory in which the concentration does not appear. In this formulation one can define, somewhat arbitrarily, all the thermodynamic machinery of a unary system, but of course second sound cannot be accounted for. The model is in no way in conflict with the phenomenological approach, but it does include the more complete dynamics of a binary system. As to Landau's theory, in spite of many similarities, there are real differences. Both have a certain amount of latitude and we have not yet found a good way to distinguish between them experimentally. So far, Landau's 2-fluid theory has been able to account for all the facts. It remains to be seen if the model can be improved to do as much while preserving its basic simplicity. If it can, then the choice between the two approaches becomes an aesthetic one. The difference in the treatment of the pressure may favor the model.



### Dynamical equations of the model

Since Landau's theory can be presented as a collection of formulas, let us list the equations of the model:

- Two equations of continuity, derived from variation of the action with respect to the velocity potentials.

- An adiabatic condition, from variation of  $T$ , one of the fundamental relations of thermodynamics. In the simplest model it fixes the temperature in terms of the specific entropies.

- Variation of the densities gives the Bernoulli equations

$$-\frac{D\vec{v}_1}{Dt} = \text{grad}\left(\frac{\partial f}{\partial \rho_1} + S_1 T\right) = 0, \quad -\frac{D\vec{v}_2}{Dt} = \text{grad}\frac{\partial f}{\partial \rho_2}.$$

Finally, at equilibrium,

$$\frac{\partial f}{\partial \rho_1} + S_1 T = \frac{\partial f}{\partial \rho_2};$$

this from variation of the densities holding  $\rho$  fixed. That is all.

### Fountain effect revisited

We promised to show that the model incorporates the fountain effect. Indeed, the last two equations inform us that, if the entropy is increased (by adding heat to the system at rest), the normal fluid will accelerate in the normal manner, while at least to begin with, the acceleration of the superfluid will be in the opposite direction. At the mouth of the capillary  $\vec{v}_1 = 0$ . At equilibrium  $\vec{v}_2 = 0$  as well, and then

$$\text{grad } p = -\rho_1 \text{grad} \frac{\partial f}{\partial \rho_1} - \rho_2 \text{grad} \frac{\partial f}{\partial \rho_2} = \rho_1 S_1 \text{grad } T.$$

At the entrance to the capillary there is a discontinuity, and

$$\Delta p = \rho_1 S_1 \Delta T.$$

This is the fountain pressure.

---

**TABLE 8.1.3**

Pressure = 0.  $s_{BC}$  from Brooks and MacDonnelly. My  $s$  is to be compared to  $\rho s_{BC}$ .

$T$	$\rho_2$	$\rho_1$	$\rho$	$\rho_1/\rho$	$\rho_2/\rho_1$	$s_{BC}$	$s_{BC}\rho$	$\rho_1/\rho_2$
.1	.14513	.17497E-8	.14513	.12056E-7	829E + 5	.68889E-5	.1E-5	1.2056E-8
.5	.14513	.11607E-5	.14513	.79980E-5	125030	.80709E-3	.117E-3	.8E-5
.8	.14513	.11607E-5	.14513	.79980E-5	1250.4	.42709E-2	.6202E-3	.8E-5
.9	.14473	.37579E-3	.14511	.25897E-2	385.145	.8084E-2		.2596E-2
1.0	.14415	.95892E-3	.14511	.66085E-2	150.32	.15562E-1	.00226	.6652E-2
1.2	.14107	.40407E-2	.14511	.27846E-1	34.912	.51689E-1	.00750	.2864E-1
1.4	.13403	.11109E-1	.14513	.76541E-1	12.0649	.13301	.0193	.8289E-1
1.6	.12090	.24307E-1	.14520	.16739	4.97407	.28431	.0413	.20105
1.8	.098205	.47147E-1	.14535	.32436	2.08299	.54155	.0757	.4801
2.0	.060925	.84682E-1	.14561	.58158	.58158	.95098	.1385	1.390
2.1	.033582	.11222	.14582	.76967	.299258	.12394E+1	.1807	.03358

Pressure = 5

$T$	$\rho_2$	$\rho_1$	$\rho$	$\rho_1/\rho$	$\rho_2/\rho_1$	$s$	$s/\rho_1$	
.1	.15284	.87476E-9	.15284	.57235E-8	$\infty$	.43073-5	3937.19	.5723E-9
.5	.15282	.95820E-6	.15283	.15282E-5	159491	.53774E-3	561.198	.627E-5
.8	.15263	.18344E-3	.15283	.12004	832.043	.37520E-2	20.4536	.1202E-2
.9	.15226	.55459E-3	.15283	.36291E-2	274.543	.79541E-2	14.3423	.3642E-2
1.0	.15148	.13510E-2	.15284	.88402E-2	112.12	.16276E-1	12.0474	.8919E-2
1.2	.14765	.52329E-2	.15287	.34228E-1	34.912	.55231E-1	10.5546	.7E-2
1.4	.13902	.13998E-1	.15296	.91479E-1	9.931420	.14263	10.1893	.1007
1.6	.12315	.30126E-1	.15313	.19655	4.08776	.30331	10.068	.2446
1.8	.095685	.57957E-1	.15345	.37697	1.65273	.57448	9.91218	.60588
2.0	.049022	.10553	.15405	.68281	.464536	1.0145	9.61338	2.1527

Pressure = 10

$T$	$\rho_2$	$\rho_1$	$\rho$	$\rho_1/\rho$	$\rho_2/\rho_1$	$s$	$s/\rho_1$
.1	.15892	.53584E-9	.15892	.33718E-8	$\infty$	.30835-5	5757.27
.5	.15891	.12284E-5	.15892	.77304E-5	12935.6	.39716E-3	323.315
.8	.15862	.29087E-3	.15892	.12056E-7	545.33	.38804E-2	14.3347
.9	.15809	.8369E-3	.15892	.12056E-7	188.9	89332E-2	10.6742
1.0	.15701	.19438E-2	.15893	.12229E-1	80.7728	.18815E-1	9.67949
1.2	.15220	.68628E-2	.15899	.43146E-1	22.1766	.62507E-1	9.10809
1.4	.14167	.17631E-1	.15911	.11068	8.035	.15848	8.98871
1.6	.12221	.37530E-1	.15938	.23494	3.25641	.33491	8.9303
1.8	.088137	.72376E-1	.15991	.45091	1.21774	.63384	8.75833
1.95	.046985	.11446E-1	.16059	.70897	.481813	.97517	8.51974
2.0	.027802	.13405	.16091	.82823	.207394	1.1252	8.39388

Pressure = 15							
$T$	$\rho_2$	$\rho_1$	$\rho$	$\rho_1/\rho$	$\rho_2/\rho_1$	$s$	$s/\rho_1$
.1	.16403	.36709E-9	.16403	.22379E-8	$\infty$	.23779-5	64777.69 .
.5	.16402	.18453E-5	.16403	.11250E-4	88887.9	.31462E-3	170.498 .
.8	.1636	.424E-3	.16403	.25833E-2	.99742	.42917E-2	170.498 .
.9	.16289	.11714E-2	.16404	.71398E-2	139.056	.10319E-1	8.80912 .
1.0	.16147	.26347E-2	.16405	.16055E-1	61.2859	.21955E-1	8.333024 .
1.2	.15540	.88647E-2	.16412	.53966E-1	17.5302	.71957E-1	8.11725 .
1.4	.14242	.22080E-1	.16429	.13415	6.45712	.17897	8.10553 .
1.6	.11839	.46810E-1	.16465	.28335	2.5292	.37664	8.04614 .
1.8	.74742E-1	.91507E-1	.16536	.55042	.816794	.71670	7.83219 .
1.95	.17795E-1	.14971	.16641	.89376	.118869		.

Pressure = 20							
$T$	$\rho_2$	$\rho_1$	$\rho$	$\rho_1/\rho$	$\rho_2/\rho_1$	$s$	$s/\rho_1$
.1	.16849	.26997E-9	.16849	.16023E-88	$\infty$	.19223-5	7120.42 .
.5	.16848	.29171E-3	.16849	.17314E-4	57755.7	.26359E-3	90.3603
.8	.16791	.59442E-3	.16849	.11250E-4	88887.9	.49635E-3	170.498
.9	.16695	.15873E-2	.16850	.11250E-4	88887.9	.1216E-1	170.498
1.0	.16512	.34758E-2	.16852	.20616E-1	47.506	.25837E-1	7.4334
1.2	.15750	.11308E-1	.1861	.66985E-1	13.9287	.83426E-1	7.37761
1.4	.14159	.27633E-1	.16883	.16329	5.12395	.20455	7.40238
1.6	.11154	.58433E-1	.16931	.34378	1.90884	.42826	7.32908
1.8	.54659E-1	.11660	.17028	.68084	.816794	.82223	7.05172

Pressure = 25							
$T$	$\rho_2$	$\rho_1$	$\rho$	$\rho_1/\rho$	$\rho_2/\rho_1$	$s$	$s/\rho_1$
.1	.17246	.22085E-9	.17246	.13841E-8	.78E - 9	.16053-5	7268.73
.5	.17245	.18453E-5	.17246	.11250E-4	88885	.23324E-3	170.498
.8	.17425	.83354E-3	.17247	.27984E-4	209	.59974E-3	48.32898
.9	.17166	.21514E-2	.17248	.11250E-4	88887.9	.14672E-3	170.498
1.0	.16803	.45751E-2	.17251	.26506E-1	36.7273	.30896E-1	6.75308
1.2	.15847	.14396E-1	.17262	.83280E-1	11.0077	.97621E-1	6.78112
1.4	.13868	.34682E-1	.17291	.10006	8.994	.23620	6.81045
1.6	.99891E-1	.74342E-1	.17356	.42668	1.34368	.49733	6.668976
1.8	.17527E-1	.15821	.17512	.90026	.11079	.99186	6.26926

Table 1.8.3. Experimental data from Brooks and McConnellly. Corrected only for  $p = 0$ .