April 21, 2021

Hydrodynamic sources for Gravitational Waves

Christian Fronsdal

Dept. Physics.Astronomy, University of California Los Angeles USA

ABSTRACT. The discovery of gravitational waves has confirmed that Gravitation is integral part of, Particle Physics, a canonical field theory that must be quantized. Recently, there is a (relativistic) Noetherian field theory, a new hydrodynamics, that allows for general vortex flows, with antecedents in the theory of superfluids. In the context of classical hydrodynamics it is unique. This paper includes a brief introduction to the theory. The non relativistic theory incorporates the phonons and rotons of Landau's theory of superfluids. The Roton has been identified with the massless and spinless Notoph of the 2-form gauge theory of Ogievetskij and Polubarinov. Highlights:

1. The role of stress and thermodynamic rupture of metastable states in common fluids extends to astrophysics where it is the origin of the rupture that may characterize some supernovas. The emission of a gravitational wave of helicity 2 is sometimes accompanied by notophs (helicity 0) that carries most or all of the recoil momentum. It is proposed to carry out another series of experiments to look for these massless waves with helicity 0.

2. The extra velocity field, the Roton, is manifest in many applications of hydrodynamics. It plays a crucial role in the analysis of rotational motion, fluid strain and vorticity, and in gravitational radiation. It has a negative kinematic potential that can be interpreted as as negative pressure.

3. With General Relativity now involved with Hydrodynamics it becomes imperative to quantize both together and to recognize the basic role that is played by the mass density in hydrodynamics. The failure to include a conserved mass density field has frustrated all attempts to find hydrodynamic sources of a hydrodynamic nature. This work provides realistic hydrodynamical sources for Einstein's equation of the General Theory of Relativity. It is based on a **compressible, relativistic fluid** that is defined by an action principle and includes vorticity, with the rotational motion that is needed for gravitational waves, both continuous and transients. The sources are derived from a Lagrangian, they satisfying the Bianchi constraints and mass current conservation. The non relativistic limit of the theory has found applications to aerodynamic lift, capillarity, metastable configurations and superfluids. It is an alternative to mechanical models of orbiting bodies treated as particles; and easier to use. It is a long sought generalization of the action principle for the irrotational motion of fluids, discovered by Lagrange in 1760. e-mail: fronsdal@physics.ucla.edu

TABLE

I. Introduction. *Hydrodynamics and Thermodynamis, Gravitation. Quantum theory and Particle Physics*

II. A brief review of important action principles. *The first action for Hydrodynamics. Thermodynamics*

III. Einstein, Hilbert and Noether. *Noether's second theorem*

IV. The field equations. *The Bianchi identity. Implications of the Bianchi identity. Conclusions*

V. The basic sources

VI. Sources for Gravitational waves. *Two scenarios. Types of radiation. Classical fields and quantum theory. The quasi-static initial state.*

VII. Time dependent sources. *Elements of the gauge theory. Time dependent initial state. The angular momentum. Radiation of gravitational wave accompanied by Notoph. Summary*

VIII. Conclusions. *Suggested experiment. Suggested investigations. Apropos the density. Concluding remarks.*

IX. Discussion. *Planned investigations. Quantum gravity*

I. Introduction

There are unmet challenges, in each of three closely intertwined areas of theoretical physics.

Hydrodynamic and Thermodynamicss

The most important tool in the theory of flows is the Navier - Stokes equation. It is based on an invention by Cauchy, the stress energy tensor. The main idea is that this tensor is divergence-less, a property related to conservation laws. But supplementary equations have always been needed, including expressions for energy density, kinetic potential and angular momentum. For complicated systems the lack of guidance becomes serious and all predictive power is lost. The practice of using formulas out of context, especially equations that can be justified only in the case of irrotational flows, does not instill confidence. We have only one reliable guide for choosing these expressions: well defined, action principles with wide fields of applications.

We do not dwell on the well known shortcomings of hydrodynamics, but simply add that, until now, there are no hydrodynamic models that include suitable sources for gravitational waves. Please see Sections 5-8 for an application of the new hydrodynamics to that problem.

A beautiful theory was devoloped during the 19'th century, of one-component homogeneous fluids at thermodynamic equilibrium. But so far there was no uniform approach to binaries and more complicated systems, or to metastable states. The common approach has been to investigate one phenomenon at a time, applying a different method for each, without the assurance of overall consistency that would come with an Action.

Gravitation

To include Gravitation among physical theories in difficulty may seem ludicrous when seen against the backdrop of unprecedented progress that is represented by the discovery of gravitational waves. See Abbott *et al* (2016, 2017), Barish and Weiss (1999). But in fact, the new discoveries makes it even more urgent to do something about the theoretical problems that remain open.

So far, material sources in General Relativity have been modelled on particles, and for good reasons, since a relativistic theory of fluids with vorticity has not been available until now.

General Relativity was created between 1905 and 1915. A milestone was reached in 1915 with the discovery of an action for the metric field. Unlike Hilbert and others, Einstein was not strongly motivated by conservation laws; but the equations are complicated, while the action is a simple all-embracing idea, so this became the heart of the theory, at first, and again today. Einstein (1918) was the first to apply the theory to gravitational waves.

The action is

$$A[g] = k \int d^4x \sqrt{-g} R[g], \quad g := \frac{1}{c^2} \det(g_{\mu\nu}) \tag{1}$$

where $R[g]$ is the Riemannian curvature scalar of the metric and k is a constant. Variation of this action with respect to the metric gives

$$\delta A[g] = k \int d^4x \delta(\sqrt{-g} R) = k \int d^4x \sqrt{-g}\ \delta g_{\mu\nu} G^{\mu\nu}; \tag{2}$$

this defines the Einstein tensor ($G^{\mu\nu}$) and the field equation: $G_{\mu\nu} = 0$.

General Relativity becomes a theory of gravitation when a source ($S_{\mu\nu}$) is included:

$$kG_{\mu\nu} = S_{\mu\nu}.$$

This is where Noether's work comes in (Noether 1917), leading to sources of gravity and to substantial progress in 3 major areas of theoretical physics.

Much important work has been done in all of these fields but here the focus is on **action principles for hydrodynamics**. We suggest taking modern particle physics as a model, and ask if there is a chance of emulating the success of this discipline, with its **amazing power of prediction**. We suggest that the approach of the 20'th century has been too easy going: fundamental principles have not been pursued with the degree of emphasis that has characterized Particle Physics. We insist that the interaction of the metric with matter fields must be a Noetherian field theory and we show that this investment pays large benefits in gravitation and in other fields, most dramatically in hydrodynamics where we discover a Noetherian field theory that includes flows with vorticity and predicts internal stresses in compressible fluids – and in the treatment of gravitational waves.

The most recent applications of General Relativity fully embrace the precept that an Action is needed, but they are of necessity limited to mechanical models of orbiting bodies treated as point-like particles. Recent hydrodynamical models are easier to use and they incorporate thermodynamical aspects such as fluid tension and metastable configurations.

Progress in gravitational fluid dynamics has an important and immediate application in the non-relativistic limit that is hydrodynamics. Conversely, a formulation of hydrodynamics as an action principle points the way to a relativistic theory of gravitation. The difficulties that are encountered have been the same in both fields: how to deal with rotational flows.

Hilbert was concerned about symmetries and conservation laws and, with the discovery of an action that is invariant under all coordinate transformations the idea that a conserved energy momentum tensor would be a suitable source for Einstein's equations came naturally. Hilbert turned the problem of interpretation over to Noether, an algebraist, who made a major discovery (Noether 1918):

The natural context for a conserved energy momentum tensor is a field theory, defined by an Action constructed from scalar fields and a Riemannian metric (Noether 1918).

In the years following this great moment in the history of physics many leading scientists turned their creative impetus to the development of quantum theory, while progress in the theory of gravitation made a slow start. This is clearly documented by Misner, Thorne and Wheeler (1972) in their influential textbook. We shall review this development in Section 2.

Quantum Theory and Particle Physics

Let us digress for a moment to review the meteoric advance of what is today referred to as Particle Physics.

A feature that characterized quantum theory from the beginning was that it correlates experimental facts with amazing efficiency: a single parameter and Ohm's law, in the context of Bohr's atomic model of 1905, was enough to give (that is, predict) a detailed, numerical account of atomic spectra. Dirac (1925) added a complete axiomatic structure that enabled Heisenberg and Pauli (1929) to create the first quantum field theory. In turn, this led to the need for rigorous internal consistency and, eventually, to an understanding of ultraviolet renormalization, made possible by a return to Hamiltonian mechanics with its action principle and canonical variables. Faddeev and Popov (1967) took the same path to create perturbative quantum gravity.

The point is that **the amazing power of prediction of quantum theories is strongly related to their formulation as action principles and to a rigid attention to mathematical consistency.** To ignore 'theoretical' or aesthetic requirements is to risk the loss of predictive power.

During the development of quantum theory there was an acute awareness of the importance of conservation laws and of the essential role that was played by an action principle with a wide field of applications.

II. A brief review of important action principles

The first to formulate a principle of maxima or minima in hydrodynamics was Daniel Bernoulli (1738). The mathematical formulation was developed by Euler and Lagrange and led to Lagrangian and Hamiltonian mechanics. A most important contribution was made by Maupertui (1741, 1747) who invented the Action and who was the first to formulate a dynamical variational principle, applicable to systems in motion.

Lagrange was aware of the practical importance of changing coordinates, and impressed by the amount of labor that may be involved. He looked for a reformulation of Newton's equations that would involve quantities with simpler transformation properties. The Lagrangian solves this problem admirably, for it is a scalar function with respect to coordinate transformations. It contains velocities but no accelerations, and yet it defines the dynamics. Today the way to deal with the Coriolis force in teaching begins by establishing the expression for the Lagrangian in terms of moving coordinates.

But there was a more important advantage. As Poincaré wrote:

"*We cannot content ourselves with formulas simply juxtaposed which agree only by a happy chance; it is necessary that these formulas come as it were to interpenetrate one another. The mind will not be satisfied until it believes itself to grasp the reason of this agreement, to the point of having the illusion that it could have foreseen this.*" (Poincaré 1908)

In Lagrange's formulation of mechanics we are not confronted with a set of equations that agree by happy chance, but by a single function, and a concise statement to the effect that actual motions minimize the action. From this compact statement all the dynamical concepts and equations are derived. And the general experience is that the set of equations derived from an action principle have a good chance of being internally consistent.

The first Action for Hydrodynamics

The variables of classical hydrodynamics are a scalar density ρ and a velocity $\vec{v}$. The fundamental equations are the equation of continuity

$$\dot{\rho} + \vec{\bigtriangledown} \cdot (\rho \vec{v}) = 0 \tag{3}$$

and an equation attributed to Bernoulli (1738),

$$\dot{\Phi} - \vec{\bigtriangledown}\Phi^2/2 - \frac{\delta W[\rho]}{\delta \rho} = 0. \tag{4}$$

Lagrange (1760) united both equations in an action principle, with the action

$$A_{NR}[\rho, \Phi] = \int dt d^3x \mathcal{L}, \quad \mathcal{L} := \rho(\dot{\Phi} - \vec{\bigtriangledown}\Phi^2/2 - \varphi) - W[\rho]. \tag{5}$$

In this theory the velocity field is restricted to the form of a gradient,

$$\vec{v} := -\vec{\bigtriangledown}\Phi.$$

There is one pair of canonical field variables, the density ρ and the velocity potential Φ. Gravity is represented by Newton's gravitational potential φ; at this point an external field. The two Euler-Lagrange equations are the equation of continuity, Eq.(3) from variation of Φ, and the Bernoulli equation, Eq. (4) from variation of ρ.

The two equations (3) and (4) make up the core of classical and modern hydrodynamics. The action principle is evoked in some textbooks, as in Lamb (1932) and Fetter and Walecka (1980), while some do not mention it.

The strong appeal of this theory is evident in the development of the theory of aerodynamic lift. The limitations implied by $\vec{v} := -\vec{\bigtriangledown}\Phi$ are vividly illustrated in Birkhoff's review of wind tunnels (Birkhoff 1950). It soon became apparent that this restriction excludes any possibility of lift, drag and flight. (The d'Alembert paradox, 1759). Considerable success was achieved when the condition was applied only locally, but that left only a framework; there was no theory based on a variational principle that incorporated this generalization. And that is the challenge that should be our guide.

Thermodynamics

The early developments of hydrodynamics were followed by 100 years of decisive progress in thermodynamics. By mid 19'th century Massieu (1869, 1876) had concluded that a thermodynamic system at equilibrium is completely defined by any one of a set of fundamental potentials, each of which is qualified to be interpreted as an energy. This statement was underwritten by Gibbs (1878) and its general validity has not been challenged. The energy concept was deeply entrenched and the idea that the energy defines the system was becoming central to thermodynamics. Some of the greatest minds of the 19'th century made sustained attempts to discover an action principle; foremost among them Helmholtz (1883). His book was critically reviewed by Poincaré (1908) who devoted a chapter of his own book to it.

Towards the end of the 19'th century there was a strong interest in basing that theory on variational principle. Gibbs paper of 1878 helped to make minimum energy and maximum entropy the cornerstones of the theory, but neither Gibbs, nor his followers, developed the action principle that could be glimpsed under the surface of his work. Maxwell was greatly inspired by Gibbs and was looking for an action principle when he died in 1879; see Rukeyser (1942). It is instructive to examine the early attempts, for both Gibbs and Helmholtz were very close to their goal. A first part of progress could have been the action for an isolated, homogeneous system at rest,

$$A(S,P,V,T) = F(V,T) + ST + VP. \tag{6}$$

Here F is the free energy, supposedly given as a function of V and T. The entropy S and the pressure P are fixed parameters and the manifold of physical configurations is defined by the Euler-Lagrange equations

$$\frac{\partial F}{\partial T} + S = 0, \qquad \frac{\partial F}{\partial V} + P = 0.$$

The essential aspect of the expression for the action is that the 4 variables are *a priori* independent; they are the 'off shell' dynamical variables. This revelation appears to have been what did the most to excite Maxwell's enthusiasm. It inspired him to invent a 3-dimensional version of the P,T diagram and to send a wax statue representing water to Gibbs.

The localization of these relations, according to Callen (1960), and the inclusion of the kinetic energy, leads to the Lagrangian for Adiabatic Thermodynamics, a unification of Thermodynamics and Hydrodynamics (Fronsdal 2011, 2020a). The equations that define the thermodynamics of an adiabatic system with free energy density $f(S,T)$: are the Euler - Lagrange equations of a well defined action inspired by (6).

III. Einstein, Hilbert and Noether

The emergence of special relativity in 1905 captured much of the attention until we come to the pivotal discoveries of 1915-1918, when Einstein's - and Hilbert's - theory of 1915 presented physicists with the challenge of finding the sources for the dynamical metric, and Noether's treatise of 1918.

Einstein alone is credited with the creation of General Relativity, Hilbert independently discovered the action

$$A[g] = k\int d^4x\sqrt{-g}R,$$

only 5 days later. The two men enjoyed a competitive collaboration, and it was Hilbert who asked Noether to assist in unraveling the mystery of symmetry and conservation. Noether was guided by the work of Cauchy (1789 -1857), although his stress tensor was a phenomenological concept.

What concerns us here is Noether's second theorem and invariance under general coordinate transformations.

Noether's second theorem

On a Riemannian space with the metric connection, consider a theory of scalar and tensor fields defined by an action

$$A[g_{\mu\nu};\phi,\psi,...;\phi_{,\mu},\psi_{,\nu},...] = \int d^4x\sqrt{-g}\ \mathcal{L} \tag{7}$$

that is expressed in terms of scalar fields, their first order derivatives and the metric tensor, invariant under general transformations of the coordinates.

Eq. (7) implies a severe restriction that rules out higher spins, about which more later. See Misner, Thorne and Wheeler (1972). That allows the use of the metric connection, in which case the covariant derivatives of the metric are zero; that is the connection that will be used from now on.

Theorem (Noether 1918). Suppose that the fields $\phi,\psi,...$ satisfy the Euler-Lagrange equations of an action with Lagrangian density $\mathcal{L}$, in Eq. (7). Suppose further that the group generated by ξ is a symmetry of the metric; that is , $\xi g = 0$. (See Eq. (17).) Then the tensor field with components

$$T_\mu{}^\nu := \sum_\phi \phi_\mu\frac{\partial\mathcal{L}}{\partial\phi_\nu} - \delta_\mu^\nu\mathcal{L} \tag{8}$$

satisfies the covariant divergence condition

$$(T^{\mu\nu})_{;\nu} = 0. \tag{9}$$

Proof. The central fact is that the action is invariant under general coordinate transformations, hence for any smooth vector field ξ the quantity

$$\mathcal{M}[\xi,\mathcal{L}] := \int d^4x\sqrt{-g}\xi\mathcal{L}, \qquad \xi = \xi^\mu\partial_\mu, \tag{10}$$

can be evaluated in terms of the action of ξ on the fields and on their derivatives. (Remember that $\xi g = 0$.) By inspection and partial integration Eq. (10) can be reduced, by use of the Euler-Lagrange equations, to

$$\int d^4x(\sqrt{-g}\xi^\mu T_\mu{}^\nu)_{,\nu} = \int d^4x\sqrt{-g}(\xi^\mu T_\mu{}^\nu)_{;\nu} = 0. \quad \text{QED.}$$

This is a surface integral, and that is what gives us the principal conservation theorems. An efficient calculation starts by interpreting the derivatives of the scalar matter fields as covariant derivatives. The covariant derivatives act as derivations on the tensor algebra this greatly simplifies the labor.

The lack of uniqueness of the conserved tensor defined by (9) alone is notorious. It is a general experience that this ambiguity is of no physical significance and that there is one that is symmetric. More important, although the expression (8) is not valid in theories that have vector fields, the Noetherian calculation based on the integral $\mathcal{M}$ in Eq. (10) is valid in gauge theories as well, including the 2-form gauge theory of Ogievetskij and Polumarinov that is prominent below. The conserved, total angular momentum in this theory (orbital angular momentum plus spin), was calculated this way (Fronsdal 2020c). We illustrate the statement in the simpler case of the relativistic action of Lagrange's theory (no spin in this case),

$$A[\rho,\psi] = \int d^4x\sqrt{-g}\mathcal{L}[\rho,\psi],$$

$$\mathcal{L}[\rho,\psi] = \frac{\rho}{2}(g^{\mu\nu}\psi_\mu\psi_\nu - c^2) - W[\rho]. \tag{11}$$

The action (5) - the Newtonian potential φ included - is the non relativistic limit of (11), with (Fronsdal 2007a)

$$\psi = c^2t + \Phi + O(1/c^2), \qquad g_{00} = c^2 + 2\varphi + O(1/c^2); \tag{12}$$

the other components Lorentzian. The Noetherian energy momentum tensor derived from (11) is **exactly**

$$T_{\mu\nu}[\rho,\psi] = \rho\psi_\mu\psi_\nu - g_{\mu\nu}\mathcal{L}. \tag{13}$$

It is well known that the condition (9) is an integrability condition for any source of Einstein's equation. The use of the energy momentum tensor (13) as a hydrodynamical source for gravity was proposed in 2007.

Remark. We call attention to the factor ρ, interpreted as a mass density. It plays the same role as the imprimitivity in electrodynamics and it is an essential dynamical variable in hydrodynamics. It is just as important in field theories that provide the sources of Einstein's equations. It leads to a new family of relativistic field theories, including gauge theories, that will need to be included in quantum gravity. In empty space ρ is a constant and this brings us back to the familiar relativistic field theories of Particle Physics.

Electrodynamics was integrated with General Relativity from the start. but any attempts to mix General Relativity with the Klein - Gordon field theory led nowhere. Perhaps the last time that it was tried was Weinberg's introduction of 'quintescence' in cosmology (Weinberg 1989). Feynman (1963) used it as a model source for gravitational waves, but the context was propagation in empty space.

IV. The field equations

Variation of the metric action, $A[g] := k\int d^4x\sqrt{-g}R$ with respect to the metric,

$$\delta A]g] = k\int d^4x\sqrt{-g}\,\delta g_{\mu\nu}G^{\mu\nu}, \tag{14}$$

defines the Einstein tensor

$$G^{\mu\nu} = \frac{\delta R}{\delta g_{\mu\nu}} - \frac{1}{2}g^{\mu\nu}R$$

and gives the field equations

$$G^{\mu\nu} = 0. \quad \textit{Einstein's equation}\text{ in vacuo} \tag{15}$$

it characterizes the metric field in a space time that is empty ('vacuum') except for the metric itself.

The discovery of field equations for the gravitational metric **in empty space** was a milestone in the development of a Theory of Gravitation. It has received a direct confirmation only recently, with the discovery of traveling gravitational waves (LIGO 2016). But it is not yet a theory of gravitation, and it is not an adequate generalization of hydrodynamics. Just as the mass density ρ appears as a source for the potential in Newton's theory (Poisson's equation), we need to add sources to the right hand side of Eq. (15).

The Bianchi identity

Theorem. The Einstein tensor, defined by Eq.(12) (14), satisfies the following equation,

$$D_\nu G^{\mu\nu} = 0, \qquad \textit{Bianchi identity}. \tag{16}$$

The operator D_ν is the covariant derivative, with the metric connection.

Proof. The action (13) is invariant under infinitesimal coordinate transformations,

$$\delta x^\mu = \xi^\mu, \qquad \delta g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu. \tag{17}$$

Hence, by Eq. (16), the variation

$$\delta A[g] = 2k \int d^4x \sqrt{-g} (\mathcal{D}_\mu \xi_\nu) G^{\mu\nu}$$

is identically zero. An integration by parts gives

$$\int d^4x \sqrt{-g}`\ \xi_\nu D_\mu G^{\mu\nu} = 0.$$

Since the vector field ξ is arbitrary this validates the statement (16).

Implications of the Bianchi identity

Let us add a source to Einstein's equation (15):

$$kG_{\mu\nu} = S_{\mu\nu}. \qquad \textit{Einstein's equation} \text{ cum fontibus} \tag{18}$$

Taking the covariant divergence we find

$$kD_\nu G^{\mu\nu} = D_\nu S^{\mu\nu} = 0. \tag{19}$$

The first expression is identically zero; therefore an inevitable consequence is that **there can be no solution of Eq. (18) for any source unless**, by virtue of the field equations of the matter fields,

$$D_\nu S^{\mu\nu} = 0. \qquad \textit{The Bianchi constraint}. \tag{20}$$

Eq.(18) is a condition of integrability and this equation is what brings the theory into contact with Cauchy's stress tensor. At first, this condition was treated with due respect and led to the first examples of interactive gravitation field theory, among the most important, a theory of the metric tensor field interacting with the vector field of electromagnetism, Einstein - Maxwell theory. This is where Emmy Noether entered history.

We have seen that, if a matter action of the type (6) is constructed from the metric and from scalar matter fields only, then there is a tensor field

$$T_\mu{}^\nu = \sum_\phi \phi_\mu \frac{\partial \mathcal{L}}{\partial \phi_\nu} - \delta^\nu_\mu \mathcal{L} \tag{21}$$

that satisfies the 'conservation law' (9) on shell. The first idea that comes to mind is to identify $S_{\mu\nu}$ with the energy momentum tensor of a Noetherian field theory, this being a solution of the Bianchi constraint.

Example, irrotational flow. This is based on the total action

$$A^[g, \rho, \psi] = A[g] + A[\rho, \psi], \tag{22}$$

where $A[\rho, \psi]$ is the matter action defined by (11). The source

$$S_{\mu\nu}[\rho, \psi] = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} A[\rho, \psi] = \frac{1}{2}(\rho \psi_\mu \psi_\nu - g_{\mu\nu} \mathcal{L}) \tag{23}$$

coincides with the Noether tensor (13) (up to the factor 1/2).

Example, Electromagnetism. If we try to extend this success to invent a coupling between the metric and electromagnetism then we find that in that case $T_{\mu\nu}$ in (21) is not symmetric, while $S_{\mu\nu}$ clearly is - see Eq. (18). Misner, Thorne and Wheeler (1972) turned away from electromagnetism at this point but it is well known that the conserved tensor of Maxwell's theory can be modified to make it both symmetric and conserved.

Let us explore the electromagnetic - metric action

$$A[g, F] = A[g] + A[F],$$

with

$$A[F] = \int d^4x \sqrt{-g} g^{\mu\nu} g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu}.$$

The field equation is

$$kG_{\mu\nu} = S_{\mu\nu} := \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} A[F].$$

That is,

$$S_{\mu\nu} = 2g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{2} g_{\mu\nu} F^2.$$

This tensor is not proportional to the Cauchy-Noether tensor, but it is evidently symmetric, and **it is conserved**.

Proof. Calculate the variation of the action $A[g,F]$ under a transformation $\delta x^\mu = \xi^\mu$. On shell; that is, by virtue of the matter field equations it is,

$$\delta A[g,F] = \int d^4x\sqrt{-g}\Big(\delta g_{\mu\nu}S^{\mu\nu} - \xi_\mu D_\nu T^{\mu\nu}\Big).$$

As both terms in the total action are seperately invariant both terms in this expression are zero. Explicitly, on shell,

$$(S_\mu{}^\nu)_{;\nu} = F^{\alpha\beta}(dF)_{\alpha\beta\mu} = 0,$$

$$(T_\mu{}^\nu)_{:\nu} = 2F^{\alpha\beta}F_{\alpha\beta;\mu} - \partial_\mu F^2 = 0.$$

Conclusion

A class of relativistic field theories, the **Noetherian field theories** defined in Eq. (7), provide sources for Einstein's field equation. The relevance of these field theories for non relativistic hydrodynamics and for the interpretation of gravitational waves was emphasized. A paralell account of Fermionic sources is not undertaken here. The source ($S_{\mu\nu}$) of Einstein's equations must be derived from a relativistic action $\mathcal{L}_{Matter}$; it does not always coincide with Cauchy's energy momentum tensor ($T_{\mu\nu}$). The conservation laws can be derived by the evaluation of the quantity $\mathcal{M}[\xi]$ in Eq.(10); they are equivalent to the Bianchi constraints.

V. The basic sources

The matter action (11) is the simplest one possible; but being restricted to irrotational flows, and just two degrees of freedom, it is of limited use. Still, it should not be underestimated. It is the only simple model that contains the equation of continuity and the conservation of mass. And it is the only way to relate, in the field theoretic context, Einstein's metric theory to Newtonian gravity, with the eponymous term $\rho\varphi$ in the Hamiltonian density. For either of these reasons any interesting generalization must include the canonical pair (ρ, Φ) of field variables. We can generalize it by adding the actions of any number of similar, spin-less matter field theories. What about field theories with vector fields?

To find a source for Einstein's equation it is enough to select an invariant matter field theory defined by an action that is constructed from the matter fields and the metric. Variation of the total action yields the field equations in the form $G_{\mu\nu} = S_{\mu\nu}$. This source satisfies the Bianchi constraints.

For simple hydrodynamics, with 4 degrees of freedom, the choices (restrcted to bosonic matter fields) are:

Type 1. The action for the spinless field is $A[\rho,\psi]$ in Eq. (11). The non relativistic limit is irrotational hydrodynamics. It can be mixed with additional scalar fields, and with electromagnetism. Alternatively, it can be made to interact with eletromagnetic fields by using the conserved current. The velocity of mass flow is irrotational in this model.

Type 2. The relativistic 2-form field $Y = (Y_{\mu\nu})$ with the action

$$A[Y] = -\frac{c^2}{12}\int d^4x\sqrt{-g}\; dY^2 \tag{24}$$

was analyzed by Ogievetskij and Polubarinov (1962, 1964, 1966), in the special case of uniform density. It can be generalized and mixed with electromagnetism,

$$A[\rho, Y, F] = -\int d^4x\Big(\sqrt{-g}(\frac{c^2}{12}\rho dY^2 + \epsilon F^2) + \gamma dY dF\Big).$$

The last term, with γ constant, makes the photon massive. It does not have an equation of continuity and no conserved mass flow. It cannot replace the irrotational model, but it can supplement it. This field theory, as well as its non relativistic limit, was used by Rasetti and Regge (1973), and by Lund and Regge (1976), in their work on vortices in String Theory and in Superfluid Hydrodynamics - where the two-form field was first identified with Landau's Roton. The 2-form is the Kalb-Ramond field of string theory and the last term is the Green-Schwartz term that cancels the anomalies in some string theories.

The two-form is a gauge field with only one propagating mode. Since this model includes a vector field the only sure way to find the source in the gravitational field equations is by invoking the Euler-Lagrange equations.

Type 3. Conservative Hydrodynamics. Much more versatile is a combination of Type 1 and Type 2, including an important mixing term,

$$\begin{aligned}\sqrt{-g}\mathcal{L}_{Matter} &= \sqrt{-g}\mathcal{L}[\rho,\psi,Y]\\ &= \sqrt{-g}\mathcal{L}[\rho,\psi] + \sqrt{-g}\mathcal{L}[\rho,Y] + \frac{\kappa}{2}\rho dY d\psi,\end{aligned} \tag{25}$$

the first term as in Eq.(11) and the second term as in (24):

$$\sqrt{-g}\mathcal{L}_{Matter} = \sqrt{-g}\frac{\rho}{2}\Big((g^{\mu\nu}\psi_\mu\psi_\nu - c^2) - f - sT\Big)$$

$$-\sqrt{-g}\frac{c^2}{12}\rho g^{\alpha\mu}g^{\beta\nu}g^{\gamma\lambda}dY_{a\beta\gamma}dY_{\mu\nu\lambda} + \frac{\kappa}{2}\rho dY d\psi. \tag{26}$$

(Notation: $dY_{\mu\nu\lambda} := Y_{\mu\nu,\lambda}+$ two cyclic permutations of the indices.) The factor ρ is needed in both terms to get the correct non-relativistic limit.

In (26) we have replaced the hydrodynamic potential $W[\rho]$ by the thermodynamic density $f + sT$; it is the local form of the expression $F + ST$ in Eq.(6). The potential $W[\rho]$ is recovered by using the thermodynamical relation $df/dT + s = 0$ to eliminate the temperature (for a constant entropy density s).

Note the inclusion of the factor ρ. The interaction term, the last in Eq.s (25) and (26), is linear in time derivatives; therefore there is no corresponding term in the Hamiltonian. The "topological" interaction term is ineffective in the special case that the density is uniform, as in empty space.

Non-relativistic Hydrodynamic is characterized by the constraint $\vec{\bigtriangledown} \cdot \vec{X} = 0$ and the Lorentzian metric, but in the present context of Relativistic Newtonian Hydrodynamics this scalar field is the wave function of the Notoph particle. In the approximation in which all the other components of the metric are Lorentzian, Eq. (26) becomes

$$\mathcal{L}[\rho, \Phi, X] = \rho(\dot{\Phi} - \vec{\bigtriangledown}\Phi^2/2 - \varphi) - f - sT$$

$$+\frac{\rho}{2}(c^2 g^{00})\Big((\dot{\vec{X}})^2 - c^2(\vec{\bigtriangledown} \cdot \vec{X}^2)\Big) + \kappa\dot{\rho}\vec{X} \cdot \vec{\bigtriangledown}\Phi + \kappa c^2 \rho \vec{\bigtriangledown} \cdot \vec{X}. \tag{27}$$

The vector field $\vec{X}^i = \epsilon^{ijk}Y_{jk}/2$; the components Y_{0i} are zero in the physical gauge. The last two terms come from the expansion (12) of ψ. When $\vec{\bigtriangledown} \cdot \dot{\vec{X}} = 0$ and $c^2 g^{00} = 1$ this reduces to the familiar, non-relativistic Lagrangian

$$\mathcal{L}_{NR}[\rho, \Phi, X] = \rho(\dot{\Phi} - K - \varphi) - f - sT, \tag{28}$$

where K is the kinetic potential

$$K := \vec{\bigtriangledown}\Phi^2/2 - \dot{\vec{X}}^2/2 - \kappa\dot{\vec{X}} \cdot \vec{\bigtriangledown}\Phi. \tag{29}$$

Electromagnetic fields have been omitted. The coupling term (the κ-term) is essential, for it is the origin of vorticity and spin-orbit coupling, needed to explain recent observations of superfluids (Kim and Chan 2004, Hall and Vinen 20.) This is a model of compressible hydrodynamics with an equation of continuity, vorticity (if the constant $\kappa \neq 0$) and internal stress.

Remark. The relativistic gauge theory has two pairs of conjugate, dynamical degrees of freedom. But in the non-relativistic sector the wave function $\vec{\bigtriangledown}\cdot\vec{X}$ of the notoph is zero, so there is only one. That is why the roton field is often fixed in applications to rotating superfluids (Fetter 1980, Hall and Vinen (1956).

The source provided for Einstein's equation by this model is the sum of the irrotational part $S[\rho,\psi]$, Eq. (23) and (27),

$$S[\rho,Y]_{\alpha\beta}=\frac{c^2}{4}\rho g^{\mu\nu}g^{\lambda\rho}dY_{\mu\lambda\alpha}Y_{\nu\rho\beta}-\frac{1}{2}g_{\alpha\beta}\mathcal{L}'[\rho,Y]. \tag{30}$$

The "topological" coupling term makes no contribution to the source; $\mathcal{L}'$ is the Lagrangian density with the topological term omitted.

In the physical gauge, where $Y_{0i}=0$, and $g_{i0}=0$, it gives

$$S_{ij}[Y]=\frac{c^2g^{00}}{2}\rho g^{kl}Y_{ik,0}Y_{jl,0}+\frac{c^2}{2}\rho g^{kb}g^{lc}Y_{ikl}Y_{jbc}-\frac{1}{2}g_{ij}\mathcal{L}',$$

We define the 3-tensor (h^{ij})

$$g^{ij}=g^{ij}_{Lor.}-h^{ij}.$$

In the approximation where h^{ij} is neglected and g^{00} is replaced by c^2, the relevant part of the total source is just

$$2S_{ij}[\rho,\psi,Y]=\rho\psi_i\psi_j+c^2g^{00}\rho\dot{X}_i\dot{X}_j. \tag{31}$$

VI. Sources of Gravitional Waves

Two scenarios

The gravitational waves observed by Ligo and Virgo (Abbott et al 2016, 2017) appear to have been provoked by an event of very short duration, thus differing greatly from the gravitational radiation studied previously by Hulse and Taylor since 1975. The emitter of the radiation is a stellar object that may have been developing in relative isolation for several billions of years. It may be supposed that most of its history has been quasi-static: a progression through a family of equilibria. It is natural to hope that an understanding of these configurations may come within the field of competence of laboratory hydrodynamics and thermodynamics, as proposed by Homer Lane (1870).

The quasi-static history of a star, during which not much gravitational radiation is emitted, is occasionally interrupted, by violent events. Such events include at least two types.

1. After heroic calculations in post-Newtonian particle theory, it has become possible to construct detailed scenarios (templates) among which there are some that can account for the observations. As a result, it now seems plausible that there are many of these objects in the Universe, and it allows us to believe that the physics of colliding and merging Black Hole Binaries is understood.

Nevertheless, a sceptic may ask for more information about the final state and the nature of the recoil, about which we have a new suggestion: Notophs.

2. Supernovas have been recorded for about 1000 years; they are very powerful emitters. Wheeler and his followers have argued that many, perhaps most, large stars eventually become small, dark objects, such as neutron stars (Wheeler 1964). To explain the existence of supernovas it has been supposed that the star has to pass through a violent stage before it can collapse. We propose that this phase can be characterized as a release of stress.

Recent developments in the hydrodynamics of compressible fluids, in the treatment of rotational flows and in the interpretation of metastable states, can help us understand the occurrence of violent transformations of a star.

Rupture and collapse of fluids have been observed in the laboratories. In the Berthelot experiment (1850) the piston of a cylindrical pump completely filled with water is withdrawn, in a slow, quasi-static development, till the pressure turns negative. The water literally hangs on to the walls and energy is stored in the form of liquid tension that is in evidence as the gradient of the second and third terms in the kinetic potential, Eq. (29). The fluid (perhaps van der Waals type) has entered a metastable state. In this state the tension field $\vec{X}$ contributes to the Hamiltonian density,

$$h = \frac{\rho}{2}(\dot{\vec{X}}^2 + \vec{\bigtriangledown}\Phi^2) + f,$$

to the kinetic potential and to the angular momentum (the spin), all of which may be released as the metastable state ruptures. What is seen subsequently is a chaotic flow, as what remains of the kinetic energy turns to heat. The stresses are gone; they may have radiated away in the form of notophs and/or electromagnetic radiation. What is left is mostly vapor, in a configuration that may once again be governed by a van der Waals equation of state.

A similar scenario may be imagined for astrophysics. Consider an ageing star, cooling and shrinking. The kinetic energy converts to high tension, a stressed, metastable state that is comparable to a wound-up spring, and holds on until it ruptures. There is a phase of intense kinetic activity, possibly accompanied by the emission of gravitons, while the reminder, <u>having shed its tension</u>, collapses to a neutron star or a Black hole. This is the familiar picture of the formation of a neutron star that was developed by Wheeler (1964); here it is backed up by a hydrodynamical model that offers an explanation, in terms of tension, of why the star has to rupture before it can collapse. See Weinberg (1972), page 305. On a smaller scale the scenario may apply to tornadoes and hurricanes, and on a yet smaller scale to breaking ocean waves.

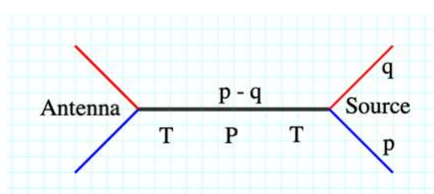
The possibility that the end of the strained star is marked by a burst of gravitational radiation, with much of the recoil taken up by the emission of notophs may explain the absence of intense and prolonged electromagnetic radiation in some cases.

Classical fields and quantum theory amplitudes

We shall calculate the gravitational radiation field associated with the source (31).

The recent discovery of gravitational waves authorizes the hope that quantum gravity shall soon be reality. Extensive work on perturbative quantum gravity has cleared all difficulties of consistency except for the divergences of loop amplitudes. It may be safe to assume that our understanding of tree diagrams is definitive; we shall assume only that our calculation of the Feynman diagram in Fig.1 will be cofirmed by the future quantum theory.

Fig.1. Feynman tree - diagram for gravitational Compton scattering. The energy nomomentum tensor T must be replaced by the source S.



In the figure, the horizontal line represents the graviton. The amplitude for the exchange of a graviton over long distances involves a source $S^{s.ce}$, a receiver S^{rec} and Feynman's propagator P:

$$S^{rec}_{\mu\nu} P^{\mu\nu,\alpha\beta} S^{s.ce}_{\alpha\beta}$$

(Feynman 1964). The source on the right is a matrix element of an operator $\hat{S}_{\alpha\beta}$ between an incoming state and an intermediate state that includes a graviton, each a state of relativistic or non-relativistic hydrodynamics.

In classical field theory one calculates the radiation field at a point $\vec{x}$ very far from the source at $\vec{x}'$,

$$h^{\mu\nu}(x) = \int d^4x' \sqrt{-g}\; P^{\mu\nu,\alpha\beta} S^{s.ce}_{\alpha\beta}(x').$$

The propagator P is a Green's function, an inverse d'Alembertian operator (Feynman 1963). The field was first calculated by Einstein (1917); in a coordinate system in which

$$g^{00} = 1/c^2, \quad g^{0i} = 0, \quad g^{ij} = -\delta_i^j - h^{ij}.$$

The result was expressed as

$$h_{ij}(\vec{x},t) = \frac{2G}{c^4}\int d^3x' \sqrt{-g}\frac{1}{|\vec{x}-\vec{x}'|} T_{ij}(\vec{x}', t - |\vec{x}-\vec{x}'|/c), \quad i,j = 1,2. \tag{32}$$

Here $\vec{x}$ is a point on the third axis. The tensor T is a projection of some energy-momentum tensor or, more generally, a suitable source.

A calculation in classical field theory evaluates the rate of transfer (for example, of energy) in terms of the time derivative of the metric field at x. Quantum theory uses another strategy. Here is a brief outline of Feynman's (1963) derivation of (32).

The action used by Feynman in his work on quantum gravity is exactly the one given in Eq. (11), including notation, but with the density $\rho = 1$. This part of the calculation relies chiefly on the Bianchi constraint. Suppose that the source is concentrated near the origin of the coordinate system and evaluate the field at a point far away, near the third axis. Then, to lowest order in h, repeated use of the Bianchi constraint in the form $k^\mu S_{\mu\nu} = 0$ allows to rewrite the effective wave amplitude. In the lowest order of perturbation theory

$$S'P(k)S = \frac{-1}{2}\frac{1}{k^2 - \omega^2 + i\epsilon}\Big(\frac{1}{2}(S_1' - S_2')(S_1 - S_2) - 2S_{12}'S_{12}\Big).$$

All the matrices are now 2-dimensional, $(S_{ij}), i,j = 1,2$. The last factor is the trace of $\tilde{S}'\tilde{S}$, where $\tilde{S}$ is the traceless part of S. Hence in Eq. (32), the matrix T has to be identified with the traceless part of the 2-by-2 matrix S.

$$2T_i^j = \tilde{S}_i^j := S_i^j - \frac{1}{2}S_k^k\,\delta_i^j, \qquad i,j = 1,2.$$

The prediction was applied with great success to the experiments by Hulse and Taylor (1975), reviewed by Weisberg and Taylor (2005). The calculation was repeated by Landau and Lifshitz (1974). The integral is over the extent of the body, very small compared to the mean distance R, so that

$$h_{ij}(\vec{x},t) = \frac{G}{c^4}\frac{1}{R}\int d^3x \sqrt{-g}\tilde{S}_{ij}(\vec{x}', t - R/c), \quad i,j = 1,2. \tag{33}$$

This is an exact expression for the radiation field, in first order perturbation theory. The point $\vec{x}$ of observation is on the third axis.

In our hydrodynamical model the source is given by

$$2S_{ij}[\rho,\psi,Y] = \rho\psi_{,i}\psi_{,j} + \rho\dot{X}_i\dot{X}_j. \tag{34}$$

Types of radiation

1. Within the program of perturbation theory we can construct stationary solutions of the field equations for the hydrodynamic fields, in the non relativistic limit, to order zero in the expansion of the metric about the Lorentzian. If such solutions are used in (33) and (34) it will result in no radiation, for radiation from any source must be accompanied by a change in the source. But if the solution is time dependent, then instead of a product $\rho\psi_{,i}(t)\psi_{,j}(t)$ the formula will have the form $\rho\psi_{,i}(t)\psi_{,j}(t')$ or $\rho\psi_{,i}(t)\dot{\psi}_{,j}(t)$. The difference between the two factors represents a change in the source, consistent with radiation.

2. Alternativelty, in the perspective of quantum field theory, one may consider any change in the source by calculation the amplitude for a transition:

$$2S_{ij}[\rho,\psi,Y] = \rho\psi_{,i}\psi'_{,j} + \rho\dot{X}_i\dot{X}'_j. \tag{35}$$

This applies to an event during which a substantial change of the emitter takes place in a short time interval.

The most spectacular event is one in which the factor $\vec{X}$, associated with the initial state, is an oscillating, stationary solution, in a state of high tension - 'the Spring'. Emission of a graviton or a coherent collection of gravitons requires a <u>recoil</u>. In view of the lack of cooperation between the two degrees of freedom (there are no $v_i\dot{X}_j$ terms) this responsibility falls on $\dot{X}'_j$; **this factor has to take a major part of the recoil; it is expected that this field stands for emitted notophs.**

The hope has been to interpret the result of the observation with enough confidence to discover the nature of the source, to understand what cataclysmic event may have caused the almost instantaneous emission of an immense amount of energy. To this end as well we must interpret the energy-momentum tensor as a current matrix element between an initial and a final configuration. And that requires some insight into a future quantum theory.

J.D. Jackson in his *Classical Electrodynamics* (1999) confronted the same question. In that case a fully fledged Quantum Electrodynamics was available and he was able to show that his treatment of radiative beta decay as a sudden appearance of electric charge, or as a gradual creation of charge over a short period of time, gave results that are consistent with QED. Lacking a fully developed quantum theory, what are our prospects?

In general terms, the interpretation of the Feynman amplitude as a matrix element of an operator $\hat{S}^{rec}P\hat{S}^{s.ce}$ means that $\hat{S}^{s.ce}$ is a product of an operator that transforms the in-state to vacuum and another that creates the out-state from the vacuum. The 'vacuum state' is the bare star, stripped of the degrees of freedom associated with the density and the two vector fields.

We must factorize the operator $\hat{S}^{s.ce}$ into a product of a creation operator and an annihilation operator. In the non relativistic limit Φ and $\rho = \delta\mathcal{L}/\delta\dot{\Phi}$ form a pair of canonical conjugate variables. The variables $\vec{X}$ and $\vec{m}$ are related by

$$\vec{m} := \frac{\delta}{\delta\dot{\vec{X}}} A[g, \vec{X}, \Phi] = \rho\dot{\vec{X}} + \kappa\vec{\bigtriangledown}\Phi. \tag{36}$$

On this basis we shall assume that the quantum field algebra is generated by operators $\hat{\rho}, \hat{\Phi}$, $\hat{X}_i$ and $\hat{\vec{m}}_i$. But $\vec{X}$ and $\vec{m}$ are not natural canonical variables and the gauge theory calls for a more complete, future treatment.

A particular choice of the incoming state does not fix the outgoing state; the amplitude assigns to each choice a probability. A likely event is that most of the recoil is taken up by a liberated Notoph (represented by the X field) a freely propagating particle. This will happen if the density of the final state tends towards uniformity for in this case the interaction term in the Lagrangian becomes an exact 4-form, an integral over the boundary, and ignorable. In other words, the effective source of the gravitational field is still localized. To support this with a higher degree of confidence we must advance the development of the quantum gauge theory.

First solution. The quasi-static initial state

The simplest choice is an isolated uniform star, rotating around the z-axis. Uniform (irrotational) rotation is described as $\vec{u} = -\vec{\bigtriangledown}\Phi$; here we postulate the simplest, circular periodic motion

$$\vec{\bigtriangledown}\Phi = \frac{a}{r^2}(-y, x, 0) = a\vec{\bigtriangledown}\phi, \quad r^2 = x^2 + y^2, \quad a = \text{ constant}, \tag{37}$$

where ϕ is the polar coordinate. Higher angular momenta may be present; that will lead to overtones on the main oscillation. Precession is another generalization.

The principal constraint of the gauge theory is $\vec{\bigtriangledown} \wedge \vec{m} = 0$; it is solved by

$$\vec{m} := \rho(\dot{\vec{X}} + \kappa\vec{\bigtriangledown}\Phi) = -\vec{\bigtriangledown}\tau.$$

or

$$\dot{\vec{X}} = -\kappa\vec{\bigtriangledown}\Phi - \frac{1}{\rho}\vec{\bigtriangledown}\tau. \tag{38}$$

In general the scalar field τ is related to the unique, propagating component, but in the stationary case there is no propagating mode.

The principal equation of motion is the equation of continuity; it is satisfied when the density is a time independent function of r alone. In a stationary state it is natural to take τ proportional to ϕ. In that case the singularity at the axis can be eliminated by the choice of normalization of τ. A special choice of ρ,

$$\frac{1}{\rho} = \frac{1}{\rho(0)} + \alpha r^2$$

makes $\Delta\vec{v} = 0$; this reduces the effect of viscosity.

All applications that have been studied previously were concerned with static configurations (*e.g* medisci, superfields) or by stationary flows (Couette flow, rotating planets). Here it is tempting to simplify the analysis by symmetry, for example by assuming that the density depends only on the cylindrical radius. That would lead to sources of the type (third components suppressed)

$$\rho \, v_i v_j \propto \rho \begin{pmatrix} -y \\ x \end{pmatrix} (-y, x) = \rho(r) \begin{pmatrix} y^2 & -xy \\ -yx & x^2 \end{pmatrix}.$$

If ρ depends only on r, then integration over the polar angle reduces this to a multiple of the unit matrix, helicity 0 and no gravitational radiation. We shall see that one way to produce the required helicity-2 waves is to allow the density to depend on the angles and on the time.

The kinetic potential takes the value

$$K = \frac{1}{2}\vec{\bigtriangledown}\Phi^2\bigg(a^2 - \kappa^2(a + \frac{b}{\rho})^2 - 2\kappa a(a + \frac{b}{\rho})\bigg),$$

often negative, and the energy density is

$$h = \frac{\rho}{2}\vec{\bigtriangledown}\Phi^2\bigg(a^2 + \kappa^2(a + \frac{b}{\rho})^2\bigg) + f.$$

If there is a phase transition at a minimum of the free energy density the energy is almost entirely converted to stress, setting the stage for rupture.

VII. Time dependent sources

Elements of the gauge theory

As a traditional gauge theory in empty space the massless two-form is well understood. As we are looking forward to developing the new relativistic field theories, with their novel density factors, we discover an unexpected simplification. As it is essential for developing the second and third solutions we begin by exploring the effect of the density on the gauge mechanism.

The wave function for the Notoph in a space with constant density is $\vec{\bigtriangledown} \cdot \vec{X}$. See Henneaux and Teitelboim (1991), Chapter 19. We shall show how this statement must be only slightly modified by the intervention of the density.

Variation of the Lagrangian (27) with respect to $\vec{X}$ gives the field equation; in a gauge in which $Y_{0i} = 0$ it is

$$\frac{d}{dt}\vec{m} - c^2\vec{\bigtriangledown}\rho(\vec{\bigtriangledown} \cdot \vec{X} + \kappa) = 0, \quad \vec{m} := \rho(\dot{\vec{X}} + \kappa\vec{\bigtriangledown}\Phi) =: -\vec{\bigtriangledown}\tau. \tag{40}$$

We represent the function τ by

$$\Delta\tau = -\frac{d}{dt}F - f, \qquad F := \rho(\vec{\bigtriangledown} \cdot \vec{X} + \kappa);$$

then the divergence of Eq.(40) reduces to $\Box F + \dot{f} = 0$,

$$\Box\rho(\vec{\bigtriangledown}\cdot\vec{X}+\kappa)+\dot{f}=0,\qquad \Box := \frac{d^2}{dt^2}-c^2\Delta. \tag{41}$$

The operator $\Box$ is the ordinary, empty space wave operator; a function that is annihilated by this operator is the wave function for a free scalar field; we choose one of them for $F = \rho(\vec{\bigtriangledown}\cdot\vec{X}+\kappa)$. (The only solution in a given context may be $F = 0$.) Then our wave equation reduces to

$$\tau = \frac{-1}{\Delta}\frac{d}{dt}F. \tag{42}$$

We conclude that, if in a given solution $\vec{\bigtriangledown}\cdot\vec{X}+\kappa = 0$, then the vector field does not play a dynamical role and no propagating Notoph is present.

The simplicity of the wave operator in (41) is welcome but surprising; the implications are not fully understood at this time.

Second solution. Time dependent initial state

The time development implied by the quasi-static evolution, such as a change of the parameters a and b with time, does not provoke gravitational radiation, but another type of time development may do it.

a. From the above accounting of helicities it is clear that what is needed is that the tensor $(\rho\dot{\vec{X}}_i\dot{\vec{X}}_j)$ contain the following helicities

$$dA\otimes dr,\quad dB\otimes dr, \tag{43}$$

where A and B are the real and imaginary parts of $(x+iy)^2$,

$$dA = 2(x,-y),\quad dB = 2(y,x)$$

and $dr = (x,y)$. Thus

$$dA\otimes dr = 2\begin{pmatrix} x\\ -y\end{pmatrix}(x,y) = 2\begin{pmatrix} x^2 & xy\\ -xy & -y^2\end{pmatrix}$$

gives, upon integration over the polar angle, the matrix $2\pi r^2\sigma_3$, and

$$dB\otimes dr = 2\begin{pmatrix} y\\ x\end{pmatrix}(x,y) = 2\begin{pmatrix} yx & y^2\\ x^2 & xy\end{pmatrix}$$

integrates to $2\pi r^2\sigma_1$.

The simplest possibilitiy is that $\dot{\vec{X}} \propto \vec{\bigtriangledown}\Im F$, where F is harmonic, $F = (x+iy)^2\exp i(\nu t+\phi)$, thus

$$\dot{\vec{X}} = \nu r\Big(b_1 dA\sin(\nu t)+b_2 dB\cos(\nu t)\Big),\qquad \vec{\bigtriangledown}\cdot\dot{\vec{X}} = 0, \tag{44}$$

where b_1 and b_2 are dimensionless constants. This will supply the factors dA and dB; the factor dr will come from the final state. We need

$$\vec{X} = \Big(b_1 dA\cos(\nu t) - b_2 dB\sin(\nu t)\Big) - \frac{\kappa}{2}(x,y);$$

the main part is $\vec{\bigtriangledown}\Re F$. The integration constant is chosen so that the Notoph field $\vec{\bigtriangledown}\cdot\vec{X}+\kappa = 0$.

b. The equation to be solved for ρ is the equation of continuity; it is obtained from the Lagrangian by variation with respect to the velocity potential Φ,

$$\frac{d}{dt}(\rho - \kappa\rho\vec{\bigtriangledown}\cdot\vec{X}) + (1+\kappa^2)\vec{\bigtriangledown}\cdot\rho\vec{\bigtriangledown}\Phi - \kappa\rho(\vec{\bigtriangledown}\cdot\dot{\vec{X}}) = 0. \quad \gamma := -(1+\kappa^2)a. \tag{45}$$

Equivalently,

$$\dot\rho + (1+\kappa^2)\vec{\bigtriangledown}\cdot\rho\vec{\bigtriangledown}\Phi = \kappa(-\dot{\vec{X}}\cdot\vec{\bigtriangledown}\rho - \dot\rho\ \vec{\bigtriangledown}\cdot\vec{X}).$$

Remark. The two terms on the right side disappear when the density is constant; which relates to the fact that the κ term in the action is then an exact 4-form.

c. We can choose the density so that the first term on the right disappears, by taking it to be proportional to $\vec{\bigtriangledown}\Re F$. Then if $\Phi = adr$ there remains only

$$\dot\rho(1+2a) = 0.$$

This solution will serve as the initial state in the rupture in which a graviton and a Notoph are emitted together. The fact that the flow has a radial, outward component component may mean that tension is in the process of building up towards rupture.

Conclusion. We have found a detailed, hydrodynamical configuration for a strained astrophysical body. This is the metastable configuration that can rupture to produce a Graviton and a Notoph.

The angular momentum

The total angular momentum density is (Fronsdal 2020c),

$$\vec{\Omega} = \vec{x}\wedge\rho\vec{v} + \vec{X}\wedge\vec{m}. \tag{46}$$

The normal development of a star is a reduction of size with a decrease in the orbital angular momentum (density $\rho\vec{x}\wedge\vec{v}$). Note that the evolutionary change in the orbital angular momentum is measurable (in principle) by observing the velocity $\vec{v}$ of mass flow. Observations of heavenly bodies suggest that the observed reduction of angular momentum cannot be accounted for by known mechanisms. For a recent review see Lagos (218)). But since only the orbital angular momentum is observed, this can perhaps be explained by taking into account the spin density $\vec{X} \wedge \vec{m}$, as in superfields (Fronsdal 2020c). During the quasi-static development the parameters a and b, and the density, may change. In particular, an increase in the density alone leads to an increase in spin (the value of b) and a decrease of the orbital angular momentum.

The forces that determine the flow and hold the star together include the gradient of the kinetic potential, as expressed by the Bernoulli equation,

$$\dot{\vec{v}} = -\vec{\bigtriangledown} K - \vec{\bigtriangledown}\varphi - \frac{1}{\rho}\vec{\bigtriangledown} p,$$

(φ is the Newtonian potential.) The complete kinetic potential is

$$K = \vec{\bigtriangledown}\Phi^2/2 - \dot{\vec{X}}^2/2 - \kappa\dot{\vec{X}} \cdot \vec{\bigtriangledown}\Phi.$$

The isobars are loci of $K + \varphi$; if the star has a surface then it is an isobar.

We have learned, from rotating fluids in the laboratory, that this may bring the system to a configuration of thermodynamic metastability and consequent rupture as the value of K approaches a critical value, perhaps near a minimum value of the free energy, as in the negative pressure experiment of Berthelot (1850).

The third solution

Radiation of gravitational wave accompanied by notophs

We have seen that, in a period before the main event the orbital angular momentum is likely to decrease even if the total angular momentum is conserved. A result is that $|K(\Phi, \dot{\vec{X}})|$ of the kinetic potential is decreasing. (It tends to be negative for a metastable state; this is a large part of the reason for the success of the theory in non-relativistic applications.) In other words, part of the orbital angular momentum is transformed to spin, or stress, the κ-term in the Lagrangian serving as spin-orbit coupling. This interferes with the balance between pressure and centrifugal acceleration. The negative part of K - the contribution of $\vec{X}$ - helps to reduce the pressure needed for the balance; that is, this part of $\vec{\bigtriangledown} K$ represents a tension that keeps the star from collapsing. This is characteristic of a metastable state; eventually a minor fluctuation triggers rupture, when the graviton and the notoph depart in opposite directions. A large portion of the angular momentum of the star is removed as helicity of the gravitons, leaving a small, massive star in a relatively relaxed state.

Recall Eq. (41), with $\dot{f} = 0$,

$$\left(\frac{d^2}{dt^2} - c^2\Delta\right)\rho(\vec{\bigtriangledown} \cdot \vec{X} + \kappa) = 0. \tag{47}$$

Very remarkably, this is a normal wave equation, without complications by the density, for the field $\rho(\vec{\bigtriangledown} \cdot \vec{X} + \kappa)$, exactly as if the density were uniform. The radial wave solution, the free Notoph field is

$$\mathcal{N} := \rho\vec{\bigtriangledown}\cdot\vec{X} + \kappa\rho = \frac{\bar{\rho}}{kR}e^{ikR-ikct}, \tag{48}$$

or rather

$$\mathcal{N} := \rho\vec{\bigtriangledown}\cdot\vec{X} + \kappa\rho = \frac{\bar{\rho}}{kR}\cos(kR-kct). \tag{49}$$

The ρ on the left is the density; $\bar{\rho}$ is a constant with the same dimensions. Next

$$\vec{\bigtriangledown}\tau = \frac{kc\bar{\rho}}{k^3}\frac{\vec{R}}{R^3}\bigg(\cos(kR-kct) + kR\sin(kR+kct)\bigg) \approx \frac{c\bar{\rho}}{k}\frac{\vec{R}}{R^2}\sin(kR-kct),$$

and finally The X - field of the emitted Notoph is

$$\dot{\vec{X}}^{\mathcal{N}}/c = \frac{\bar{\rho}}{\rho}\frac{1}{2}\frac{1}{(kR)^2}\frac{\vec{R}}{R}\bigg(kR\cos(kR-kct) - 2\sin(kR-kct)\bigg) - \frac{\kappa}{c}\vec{\bigtriangledown}\Phi \tag{50}$$

where $\bar{\rho}$ is uniform, the wave number k a constant. The last term in (50) will be ignored. To a sufficient approximation

$$\rho\dot{\vec{X}}/c = \bar{\rho}\frac{\vec{R}}{kR^2}\sin(kR-kct).$$

The initial state is identified with the second solution, the wound-up Spring,

$$\dot{\vec{X}}^{SP}/c = \nu/c\bigg(b_1 dA\sin(\nu t) + b_2 dB\cos(\nu t)\bigg). \tag{45}$$

Averaging over the azimuthal angle gives

$$\frac{1}{2\pi}\int d\phi(\frac{\rho}{c}\dot{X}_i^{\mathcal{N}}\frac{1}{c}\dot{X}_j^{Sp}) = \bar{\rho}\frac{\nu}{kc}\sin^2\theta\bigg(b_1\sin(\nu t)\sigma_1 + b_2\cos(\nu t)\bigg)\sin(kR-kct).$$

Here ν is the frequency of the Spring and kc is the frequency of the Notoph and integration over volume gives the final result

$$(h_{ij}) = 10^{-18}\frac{1}{d}\frac{M}{M_{Sun}}\nu kb\left(\sin(\nu t)\sigma_1 + \cos(\nu t)\sin(\nu t)\right). \tag{50}$$

The total participating volume M was defined by replacing the density ρ by the constant $\bar{\rho}$ and the factors $\sin\theta^2$ and $\sin(kR-kct)$ by unity. All estimates of the numerical values of this factor are tenuous.

The constants b_1 and b_2 are the unknown, dimensionless parameters that characterise the strength of $\dot{\vec{X}}$ for the initial state; it was introduced in Eq.(44). The frequency ν is that of the Spring and $\nu' := kc$ is the frequency of the Notoph wave.

Summary

We have found that, to zero order in the metric, the notoph wave function $\vec{\bigtriangledown}\cdot\vec{X}$ is a solution of the wave equation (43). To serve as a source for gravitational wave this solution must appear as the result of a transition from the second solution in which the field $\vec{X}$ is oscillating in the angular directions to the third solution in which the Notoph is emitted.

VIII. Conclusions

This paper points out that likely scenarios for observable gravitational radiation include purely hydrodynamical events, including some that end with the emission of Notophs. In view of its role in accounting for numerous phenomena in hydrodynamics, some of which had remained enigmatic over many years, the existence of this kind of field (and particle) is hardly in doubt. In the non-relativistic context the field $\vec{X}$ has been identified with fluid stress. It appears in thermodynamics when a fluid goes into a metastable configuration, leading to violent rupture. The prediction of a catastrophic event goes well with the very high density of energy emitted at the end of the emission of a burst of tgravitons.

The introduction of the 'Lagrange parameter' $\vec{X}$, as a second type of flow, including the coupling between both (the κ term), into hydrodynamics adds vorticity to the mass flow, as well as a spin-orbit coupling (the same term), which explains the ability of some superfluids to remember its angular momentum through a frozen period.

A study of cylindrical Couette flow has revealed that the vector field $\vec{X}$ has the interpretation of internal fluid stress. The connection with vorticity, already proposed by Rasetti and Regge (1973) is confirmed. Here we have tried to apply those results to the emission of gravitational waves.

A most striking feature of the theory is that the kinetic potential and the kinetic energy are not related in the familiar way, see eq. (29) ???? for the kinetic potential. . Likewise, the presence of tension means that the angular momentum is not always related to the velocity in the way that was expected expected. The gradient of K is an internal stress field, balanced by the pressure and external forces including gravity. The study of menisci and capillary action was taken below the surface and related to sonoluminescence and to metastable fluid configurations. For compressible fluids the theory yields the density and pressure profiles in the bulk; not just the shape of the surface.

Calculations of simple phenomena, such as the negative pressures discovered by Berthelot (1850), give support for the idea that the role of the stress field $\vec{X}$ can help us understand a category of metastable states in thermodynamics. (Fronsdal 2020c).

An application to rotating planets was found to predict (!) planetary rings. These simple rings appear as features of solutions of the Euler-Lagrange equations, with no extra input. It is not necessary to justify the rings by inventing a cataclysmic past history. The rings can disappear gradually and undramatically.

Suggested experiment

The Ligo-Virgo experiments were prepared to observe radiation with helicity 2. If a Notoph is involved then for every N observations of graviton there should be an equal number of absorbed notoph waves with helicity 0. Since they are emitted in opposite directions it is not certain that gravitons and notophs will be correlated in observations at one location.

Suggested investigations

The new degree of freedom included in conservative hydrodynamics has an effect on galactic velocities and on the total energy stored in the galaxies. The remarkable streams seen in the Milky Way hint at other rewarding investigations. Applications to superfluids are under way as is a study of sonoluminescence, wind tunnels (lift and drag), atmospheric condensation and oceanic waves.

Exact solutions of Einstein's equation in the presence of realistic hydrodynamic sources have so far been limited to static solutions with full rotational symmetry. Exact solution for the case of a stationary solution with stress should be feasible. A relaxation from full rotational symmetry to cylindrical symmetry implies a major complication since two independent variables are involved, but solutions with uniform rotation are possible. And perhaps, with the recent progress in solving Einstein's equations the time may have come to attack this problem.

The close relationship between Conservative Hydrodynamics (Section 5, type 3, with Landau's 2-flow theory of superfluids, and the recent interest in quantum Hydrodynamics (Section 6), suggest that a theory of neutron scattering by superfields may have become available.

Apropos the density ρ

Its inclusion in $A[\rho, \psi]$ goes back to Laplace and is clearly necessary to obtain the correct non relativistic limit. Here it is an essential novelty that was missing in early attempts to use particle field theories to construct sources for Einstein's equation. It is analogous to the classical problem of electromagnetic fields in a continuous medium. A version of electromagnetism has a dynamical permitivity (Fronsdal 2007b). It may have an application to the theory of electromagnetic fields on a background of a photon gas; or it may be used as an infrared regularizing device. Even gravity can be modified by including a density factor in the action as an infrared regularization, a prerequisite for quantization of Gravity (Weinberg 1970), (Marciano and Sirlin 1975).

The fact that the density factor is needed to relate hydrodynamics to relativistic field theories suggests that quantization of of these new field theories is an urgent problem. A simple guess is that a part of the solution, in the most basic case of the scalar field, is that the basic commutatation relation is

$$[\psi(x), \rho\psi(y)] = i\hbar\delta(x-y),$$

that the Hilbert space is generated by $\rho\psi(x)$ on the vacuum and that the operator conjugate to the operator $\psi(x)$ is $\rho\dot{\psi}(x)$. In the limit of empty space this novelty may be insignificant, but special circumstances may exist where a measurable effect may be anticipated. The case of the anomalous moment of the muon is suggestive because of the very high accuracy of the measurements as well as the large mass relative to the electron. The high rate production of muons in the recent collision experiments at CERN are observed in an environment that is a poor approximation to empty space.

The idea that ordinary, massive density extends to a constant density in empty space suggests it must be included in the total balance of matter in the universe; it may account for some or all of the missing Dark Matter. Weinberg has pointed out that the the global density needed for his cosmological "quintescence" agrees with the average density of known mass in the universe. In the same context it should also be pointed out that the energy of strain should also be taken into account.

Concluding remarks

Gravitational waves have been found to travel through empty space; like photons, they do not need a medium. The matter fields of the first category (Type 1, Section V) are fields associated with diverse kinds of matter, that propagate by oscillations with one or more densities. The notoph field is very different; it is massless and it has much in common with the metric field. And there may be only one Notoph. That is, we are suggesting that there are many phonons but, possibly, only one roton.

When analyzing gravitational waves we should be open to the possibility that they may be accompanied with waves of massless and spinless notophs.

To keep the overall picture as simple as possible it is tempting to suppose that the 2-form that is needed for vorticity is a fundamental field like gravity and electromagnetism; that there is only one, universally coupled to matter. Clearly it will make a contribution to the total mass and energy of the Universe. It interacts with matter, it manifests itself in phenomena like liquid stress that are difficult to recognize. It may make the photon locally massive and it makes a significant contribution to gravitational lensing. It will affect the orbits of stars in galaxies. In short, it is a prime candidate for Dark Matter.

Data Availability.

"The data that support the findings of this study are available from the corresponding author upon reasonable request."

References

Abbott, B.P.; *et al*, (2016). "Observation of Gravitational Waves from a Binary Black Hole Merger". Phys. Rev. Lett. 116 (6) 061-102 (2016)

Abbott, B.P. *et al*, GW190412: Observation of a binary-black-hole coalescence with asymmetric masses". *(LIGO Scientific Collaboration and Virgo Collaboration 90412 Cite as: arXiv:2004.08342 [astro-ph.HE] (2020)

Barish, B.C. and Weiss, R. "LIGO and the Detection of Gravitational Waves". Physics Today. 52 (10): 44 (1999)

Bernoulli, D., *Hydrodynamica*, Argentorat, (1738)

Berthelot, "Sur quelques phénomènes de dilatation forcé des liquides", Ann. Chim. Phys. 30 (3) 232-237 (1850)

Birkhoff *Hydrodynamics*, Dover NY (1950)

Callen, H.B., *Thermodynamics*, Wiley, New York (1960)

Dirac, P.A.M. (1925). "The Fundamental Equations of Quantum Mechanics". Proc.R. Soc. A **109** (752) 642 (1925).

Einstein, A., "The Principle of Relativity", Sitz.Preuss. Akad. Wiss. **142** (1917)

Faddeev, L.D. and Popov, V., "Feynman diagrams for the Yang-Mills field", Phys. Lett. B, 25 (1): 29, doi:10.1016/0370-2693(67) (1967)

Fetter, A.L. and Walecka, J.D., *Theoretical Mechanics of Particles and Continua*, MacGraw-Hill NY 1980.

Fetter, A.L., "Rotating trapped Bose-Einstein condensates", Rev Mod.Phys. **81**, 648- 691 (2009)

Feynman, R.P. "Quantum theory of gravitation", Acta Phys Pol **24** 841-866 (1963)

Fronsdal, C. "Ideal stars in General Relativity", Gen. Rel. Grav **39**, 1971-2000 (2007a)

Fronsdal, C., "Reissner-Nordstrom and Charged Gas Spheres", L.M.P. **82** 255-273 2007b).

Fronsdal, C., "Heat and Gravitation, the action Principle", Entropy **16**,1515-1546 (2011)

Fronsdal, C., *Adiabatic thermodynamics of Compressible Fluids, Hydrodynamics to General Relativity.*, World Scientific (2020a)

Fronsdal, C., "Stability analysis for cylindrical Couette flow of compressible fluids", Physics of Fluids (2020b)

Fronsdal, C., "Metastable Liquids, Capillary Action and Superfluids", to be published (2020c)

Fronsdal, C., "Under the Meniscus", To be published (2021)

Gibbs, J.W., "On the equilibrium of heterogeneous substances", Trans.Conn.Acad. 1878.

Heisenberg, W. and Pauli, W., "Zur Quantendynamik der Wellenfelder, Zeit. Phys. 56 (1929), 1-61 (1929)

Hall, H.E. and Vinen, W.F., The Rotation of Liquid Helium II. The Theory of Mutual Friction in Uniformly Rotating Helium II, Proc. R. Soc. Lond. A 1956 **238**, doi: doi.org/10/10/98/rspa1950.0215 (1956).

Henneaux, M and Teitelboim, C. *Quantization of Gauge systems*, Princeton U. Press (1994)

Helmholtz, H. “Die Thermodynamik chemischer Vorgänte”. In Wissensch. Abh.; Barth, Germany, Volume II, 958-978 (1883)

Hulse, R.A., and Taylor, J.H., “Discovery of a pulsar in a binar y system”, ApJ, 195-L51-53 (1975)

Henneaux and Teitleboim (1991)

Jackson, J.D., “Classical electrodynamics”, John Wiley (2001)

Kalb, M and Ramond, P., “ ”Classical direct interstring action”. Physical Review D. **9** (8): 22732284. doi:10.1103/physrevd.9.2273. (1974).

Kim, E. and Chan, M. “Probable observation of a supersolid helium phase’, Nature. 427(6971):225227. doi:10.1038/nature 02220 (2004)

Lagos, C. del P., “Angular momentum evolution of galaxies: the perspective of hydrodynamical simulations”, Proceedings IAU Symposium No. xxx, International Astronomical Union (2018)

Lagrange, J.M., Taurinensia, ii. 1760, Oeuvres, Paris (1867,1892)

Lamb, H. *Hydrodynamics*, Cambridge U. Press 1916).

Lane, J.H., “On the Theoretical temperature of the sun; under the hypothesis of a Gaseous Mass maintaining its Volume by its internal Heat, and depending on the Laws of Gases as known to Terrestrial experiment,”Am.J.Sc.&Arts, **50** 57-74 (1870)

Landau, L. and Lifshitz, page 3

Lund, F. and Regge T., “Unified Approach to strings and vortices with soliton solutions”, Phys. Rev. D. **14** 1524-1535 (1976)

Marciano, W.J. and Sirlin A. “Dimensional Regularization of Infrared Divergences”, Nuclear Physics B 88(1):86-98 (1975)

Massieu, F., “Sur les fonctions caractristiques des divers fluides”, C. R. hebd. Séances Acad. Sci., 69, pp. 858-862 and 10571061 (1869)

Massieu, F., *Mémoire sur le fonctions Charactéristiques des Divers Fluides et sur la Théorie des Vapeurs.* Acad. des Sci. de l’Institut National de France, XXII, 1-92 (1876)

Maupertuis, P.L., “Loi du repos des corps”, Acad. des Sci, Paris (1741)

Maupertuis, P.L., “Loix du mouvement et du repos”, Berlin Acad. Sci (1746)

Misner, W., Thorne K.S. and Wheeler, J.A., *Gravitation*, Princeton U. Press (1972)

Noether E., “Invariante Variationsprobleme”. Nachr. D. König. Gesellsch. D. Wiss. Zu Gttingen, Math-phys. Klasse.: 235-257 (1918) (1964)

Ogievetskii, V.I. and Polubarinov, I.V., Nuovo cimento 23, 173 (1962); JETP 45, 237 (1963). Soviet Phys. JETP 18, 166 (1964); Ann. Phys. (N.Y.) 25, 358 (1963); Nuclear Phys. 76, 677 (1966)

Peters, P. C. and Mathews,J. 1963,Phys.Rev.,131,435.

Poincaré, H. *Thermodynamique*, Gauthier-Villars, Paris 1908.
Rasetti, M and Regge, T. , “ Quantum vortices and diff (R3)”, in
Lecture Notes in Physics, Volume 20. Physica bf 80 A, 217 (1973)
Rukeyser, M., *Willard Gibbs*, Ox Bow PReSS, Woodbridge, Conn.,
especially from page 199 onwards. (1942)
Weinberg, S., *Gravitation and Cosmology*, John Wiley & Sons, New York
Weinberg, S., *Cosmology*, Oxford University Press (2008)
Weinberg, S., The cosmological constant problem”, Rev. Mod. Phys. 61, 1 (1989).
Weisberg, J.M. and Taylor, J.H. “The Relativistic Binary Pulsar B1913+16:
Thirty Years of Observations and Analysis”, APJ, **722**, 1030 (2010)
Wheeler, J. A., “Geometrodynamics and Issue of the Final State”, in *Relativity, Groups and Topology*, by D.C. De Witt and B. DeWitt,
Gordon and Breach, NY (1964)