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* Ideal Stars and General Relativity

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ABSTRACT. We study a system of differential equations that governs the distribution of matter in the theory of General Relativity. The new element in this paper is the use of a dynamical action principle that includes all the degrees of freedom, matter as well as metric. The matter lagrangian defines a relativistic version of non-viscous, isentropic hydrodynamics. The matter fields are a scalar density and a velocity potential; the conventional, four-vector velocity field is replaced by the gradient of the potential and its scale is fixed by one of the eulerian equations of motion, an innovation that significantly affects the imposition of boundary conditions. If the density is integrable at infinity, then the metric approaches the Schwarzschild metric at large distances. There are stars without boundary and with finite total mass; the metric shows rapid variation in the neighbourhood of the Schwarzschild radius and there is a very small core where a singularity indicates that the gas laws break down. For stars with boundary there emerges a new, critical relation between the radius and the gravitational mass, a consequence of the stronger boundary conditions. Tentative applications are suggested, to certain Red Giants, and to neutron stars, but the investigation reported here was limited to polytropic equations of state. Comparison with the results of Oppenheimer and Volkoff on neutron cores shows a close agreement of numerical results. However, in the model the boundary of the star is fixed uniquely by the required matching of the interior metric to the external Schwarzschild metric, which is not the case in the traditional approach. There are solutions for which the metric is very close to the Schwarzshild metric everywhere outside the horizon, where the source is concentrated. The Schwarzschild metric is interpreted as the metric of an ideal, limiting configuration of matter, not as the metric of empty space.

1. Introduction

Background

Relativistic hydrodynamics, as a dynamical theory, has not yet been fully developed. Modern textbooks on General Relativity all present the same intuitive idea of a phenomenological fusion of Classical Thermodynamics and General Relativity, often without encouraging a skeptical attitude. In this paper we shall try to formulate a dynamical theory of matter interacting with the metric, leaving the thermodynamic aspects to be studied within the framework of the theory. As it turns out, it will be necessary to challenge some traditional concepts, which we hope will open a discussion and perhaps indicate a direction of a future theory of relativistic hydrodynamics.

Our reluctance to entirely embrace some of the well known paradigms that supplement Einstein's General Relativity centers on Tolman's famous formula for the source term in Einstein's field equations, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, namely

$$T_{\mu\nu} = (\hat{\rho} + \hat{p})\, U_\mu U_\nu - g_{\mu\nu}\hat{p}.$$

(The density $\hat{\rho}$ and the pressure $\hat{p}$ must not be identified with the scalar fields ρ and p introduced below.) To apply this formula one requires additional input in the form of an equation of state that relates $\hat{p}$ to $\hat{\rho}$. And something must be done to pin down the 4-velocity field U. The question of how to do that is a central issue of this paper.

The proposal

We propose to limit the investigation to irrotational flows, introducing a local velocity potential, as is done in certain applications of non-relativistic hydrodynamics; the vector field U is replaced by Ψ, $\Psi^\mu = g^{\mu\nu}\psi_{,\nu}$. There is precedence for this, in Jeans' paper [J] of 1902. Believing that any theory gains internal coherence and plausibility by being formulated as a variational problem, we introduce a dynamical action principle, including a matter lagrangian. It is a model, too simple to be realistic, but it is a direct generalization of non-relativistic isentropic hydrodynamics; it does not violate any general physical principles, and there is nothing to suggest that it should be unsuitable for describing certain configurations of an actual physical system. It will be applied to the problem of the equilibrium of idealized spherical stars and, later, to the question of their stability with respect to radial oscillations. An application to cosmology is in preparation.

The Euler-Lagrange equations of the model include a conservation law

$$\partial_\mu J^\mu = 0, \quad J^\mu = \sqrt{-g}\,\rho\,\Psi^\mu.$$

They also impose a restriction on $g_{\mu\nu}\Psi^\mu\Psi^\nu$, alternatively derivable (in a slightly weaker form) from the (contracted) Bianchi identities and the conservation law, but they do not fix this quantity at unity. In a dynamical context we expect that all fields are governed by equations derived from the action principle and then it is not possible to impose normalization conditions in an *ad hoc* manner. Nor is it advisable to define the action principle in terms of a restricted class of variations that respect an *a priori* normalization, as was advocated in the papers [SW], [S] and [Ta]. It is of course possible to define a normalized vector field U by rescaling of the gradient field Ψ and adopting Tolman's formula for $T_{\mu\nu}$ in terms of $U, \hat{\rho}$ and $\hat{p}$. The question is whether these are the fields that are related by an equation of state.

Outline and conclusions

The model, introduced in Section 2, is based on the following matter contribution to the action of General Relativity,

$$A = \int d^4x\sqrt{-g}\Big(\frac{\rho}{2}(g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - c^2) - V[\rho]\Big) =: \int d^4x\sqrt{-g}\ \mathcal{L}\ . \tag{1.1}$$

It involves 2 scalar fields, a "density" ρ and a velocity potential ψ. The associated energy momentum tensor is used as source for Einstein's equations. The metric is limited throughout this paper to a class that is characterized by rotational symmetry and the existence of coordinates t, r, θ, ϕ such that the line element takes the form

$$(ds)^2 = \mathrm{e}^\nu(dt)^2 - \mathrm{e}^\lambda(dr)^2 - r^2d\Omega^2, \tag{1.2}$$

with $g_{tt} = \mathrm{e}^\nu$ and $g_{rr} = \mathrm{e}^\lambda$ depending only on r and t. A remark at the end of the section points out differences between the conventional approach and the point of view to which we are led by the model.

Section 3 deals with the static limit or equilibrium configuration, beginning with a radically new interpretation of the Schwarzschild metric. It is not the metric of everywhere empty space (which is Minkowski) but instead the metric of a limiting configuration of matter, concentrated at the horizon. This interpretation of the Schwarzschild metric is supported by another argument. Let us approach the Schwarzschild problem from the point of view of newtonian gravity. We know, for we teach this to our undergraduates, that a spherical shell of matter (at $r = R$) gives rise to no gravitational field on the inside. Indeed the potential is

$$\phi(r) = -\frac{MG}{r}\theta(r-R) - \frac{MG}{R}\theta(R-r). \tag{1.3}$$

We get this result by viewing the determination of the potential on the outside, respectively the inside, as two separate problems. It would not occur to us propose that the function $-MG/r$ that represents the potential on the outside, analytically continued to the region $r < R$, has anything to do with the problem, unless, of course, we were ignorant of the distribution of mass. Unfortunately, that is precisely what was done after Schwarzschild presented his solution for the outside.

Also in this section (Section 3) we introduce our choice of the functional $V[\rho]$ that will be used for all our calculations, the simple form $V[\rho] = a\rho^\gamma$, a and γ constant. There are strong internal indications that V may be interpreted as a relativistic analogue of the internal energy per unit mass, then the Lagrangian density $\mathcal{L}$ is the pressure (see [S] and [Ta]) and we obtain the isentropic or polytropic equation of state

$$p = a(\gamma - 1)\rho^\gamma.$$

The number γ would thus be interpreted as the ratio C_p/C_v of heat capacities at constant pressure and volume. This number has been taken to be constant, or nearly so, or at least piecewise constant, in all works on stellar dynamics that we are acquainted with. The choice of density to be entered into the equation of state varies (particle number density, energy density, density of free electrons ...). Thus Oppenheimer and Volkoff take over Chandrasekhar's equation of state [C] but substitute energy density for mass density [OV].

Our point of view is that taking $V[\rho] = a\rho^\gamma$ is an attractive choice for V, successful in the non-relativistic limit, and suitable for a preliminary exploration of the physical systems within the compass of the model. The interpretation of the lagrangian density with 'pressure' is natural and leads to the equation of state $p = a(\gamma - 1)\rho^\gamma$; we expect that the constant γ can be related to C_v/C_p. Thus the thermodynamical aspects of the theory are derived from within the theory itself.

Section 4 is devoted to a study of the case of weak fields. In this limit our model reduces exactly to the theory developed by Emden [Em] and Eddington [Ed], although the equations that define their theory did not originally come from an action principle. We shall raise some questions about their choice of boundary conditions.

Eddington dismisses the possibility of gaseous distributions extending to infinity because they exhibit a singularity at the center. Chandrasekhar and others have proposed to accept such singularities as a manifestation of a local breakdown of the gas laws. Eddington's real problem was that, being prevented (by the singularity) from formulating boundary conditions at the origin, he was also unable to fix the solution by imposing asymptotic conditions, this because his equations do not involve the gravitational potential ν itself, but only its derivative. In the approximation of weak fields, the only innovation that our theory brings with it is that it repairs this particular defect, making it possible to impose asymptotic boundary conditions and to make precise predictions for the singularity at the center.

The polytropic index is the number n defined by

$$\gamma = 1 + \frac{1}{n}.$$

It has been known, since the early work of Emden [Em] that the value $n = 5$ separates two qualitatively different regimes:

$0 < n < 5$: "Gas spheres" of infinite extension.

$5 < n < \infty$: Finite stars with boundary.

Section 5 studies the regime of infinite distributions, with applications to stellar atmospheres and possibly to gaseous giants such as Betelgeuze and Capella; we present results of numerical integration of the equations that determine the equilibrium in the model. The exact solutions differ from those of the weak field approximation to a degree that probably could not have been predicted. The density is strongly peaked in the neighbourhood of the Schwarzschild radius; there is a central region with low density (and lower temperature). There is also a very small central core where the Emden function p/ρ turns negative and where it is believed that the gas laws break down.

Section 6 deals with the regime of stars with boundary, spherical stars with matter confined to $r < R$. They are characterized, according to Eddington and others, by the fact that the density reaches zero at $r = R$. But this choice of boundary conditions seems to us to be dubious, and certainly not possible in the proposed model, where the boundary is determined by matching to the exterior Schwarzschild metric,

$$\nu(R) + \lambda(R) = 0, \quad \lambda(R) = \ln(1 - 2m/R).$$

The first condition determines R and the second one determines m. Eddington has only the second equation at his disposal. We apply this boundary condition and examine the solutions near the center of the star, to discover a critical relation between mass and radius.

In this paper the effect of radiation pressure (important for masses larger than that of the sun), as well as other necessary refinements, have not been taken into account. For this reason we do not discuss applications to real stars, except for a tentative discussion of neutron cores at the end.

In this paper the units are such that $G = c = 1$.

2. A matter model.

A standard model for irrotational flow describes the state of a continuous distribution of matter in terms of a density ρ and a velocity potential Φ, the velocity being the negative gradient of Φ. The equation of motion for the velocity (the hydrostatic condition), and the continuity equation for the current, are both derived from the variational principle with action ([FW] p. 304)

$$\int d^3x \Big(\rho\dot{\Phi} - \frac{\rho}{2}\vec{v}^2 - V[\rho]\Big),$$

where the internal energy density V often depends only on ρ (isentropic case) and is determined by the equation of state. The expression that is most commonly used for V contains a term linear in ρ that represents the external force and in addition a term $\rho S[\rho]$, where S is the internal energy per unit of mass. A relativistic version of this theory has two scalar fields, ρ and ψ, and the action

$$A = \int d^4x\sqrt{-g}\Big(\frac{\rho}{2}g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - W\Big) =: \int d^4x\sqrt{-g}\ \mathcal{L}\ . \tag{2.1}$$

The nonrelativistic theory in flat space is recovered with

$$\psi = c^2t + \Phi, \quad W = \frac{c^2}{2}\rho + V.$$

(Everywhere but here we have set the velocity of light equal to 1.)

The vector field Ψ defined by

$$\Psi^\alpha = g^{\alpha\beta}\psi_{,\beta}$$

can be interpreted as a four-dimensional flow velocity. The physical three-velocity $-\vec{\Psi}/\Psi^t$ is invariant under local scaling; the field ρ acts as a gauge fixing field, eliminating the unphysical, fourth degree of freedom. The traditional approach is to fix the normalization by hand. This amounts to renormalizing the density and one has to show that this renormalization qualifies the density for the role assigned to it in thermodynamics, specifically in the equation of state; once an equation of state is imposed one is no longer doing pure phenomenology. We are not aware of any reference where this question is brought up, let alone settled.

The energy momentum tensor is

$$T_{\mu\nu} = \rho\,\psi_{,\mu}\psi_{,\nu} - g_{\mu\nu}\,\mathcal{L}\ . \tag{2.2}$$

In particular, in the metric (1.2),

$$T_t^t = \mathrm{e}^{-\nu}\rho(\dot\psi)^2 - \mathcal{L},\ \ T_r^r = -\mathrm{e}^{-\lambda}\rho(\psi')^2 - \mathcal{L},\ \ T_t^r = -\mathrm{e}^{-\lambda}\rho\psi'\dot\psi\ .$$

The matter field equations are

$$\frac{1}{2}(g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - 1) = (dV/d\rho),\ \ \partial_\mu\big(\sqrt{-g}\,\rho\,g^{\mu\nu}\psi_{,\nu}\big) = 0. \tag{2.3}$$

The first equation fixes the scale; it comes from variation of the field ρ. The second equation comes from variation of the field ψ; it is a conservation law for the current $\sqrt{-g}\rho g^{\mu\nu}\psi_{,\nu}$.

A common extra ingredient in most works in this area is a conserved quantity identified as baryon number. In the model this role is taken by the field ρ. In the non-relativistic theory ρ is the mass density and ρV is the internal energy, inclusive or not of rest mass.

A consequence of (2.3) is that

$$\mathcal{L} = \rho\frac{\partial W}{\partial\rho} - W = \rho\frac{\partial V}{\partial\rho} - V,$$

which shows that the lagrangian density $\mathcal{L}$ is the pressure. ([FW] page 304.) In the case that $dV/d\rho = 0$ the first field equation gives

$$g_{\alpha\beta}\Psi^\alpha\Psi^\beta - 1 = g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} - 1 = 0\ \ (\text{when } dV/d\rho = 0). \tag{2.4}$$

This condition is appropriate only in the case that no forces other than those that of gravity are present. There will be discussion of this point in Section 3.

This model of a matter distribution in the context of General Relativity is an alternative, under the limitation to irrotational and isentropic flows, to the traditional approach first proposed by Tolman [T], and the simplest one possible. In Tolman's theory the energy momentum tensor is expressed in terms of a velocity field, *a priori* normalized, and two functions interpreted as energy density and pressure. The pressure corresponds to the function $\mathcal{L}$ and the energy density to the function $g^{00}\rho\dot\psi^2 - \mathcal{L}$. Most accounts of stellar dynamics assume that the entropy is either constant or irrelevant ([MTW], page 599) so that the thermodynamics is both irrotational and isentropic. An equation of state is imposed by hand. The existence of dynamics in the form of a variational principle is plausibly inferred from the Bianchi identities via Einstein's equation, but only the Bianchi identities themselves were invoked.

The model studied here is not yet fundamental theory, since the choice of the potential $V[\rho]$ remains open, but it goes a step beyond Tolman's theory. Once the potential has been fixed nothing but the imposition of boundary conditions remains. If the model admits solutions that account for observed phenomena then something has been learned concerning the types of matter distribution that are consistent with Einstein's equations and thus suitable for the framework of General Relativity.

We shall try to find realistic solutions of the system that consists of Einstein's equations,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$

together with the Euler-Lagrange equations (2.3). The metric will be assumed to have rotational symmetry. We assume that there are coordinates t, r, θ, ϕ such that the line element takes the form

$$(ds)^2 = \mathrm{e}^{\nu}(dt)^2 - \mathrm{e}^{\lambda}(dr)^2 - r^2 d\Omega^2, \qquad (2.5)$$

with ν and λ depending only on the coordinates t and r, and we ask that ρ and ψ also depend only on t and r; there are then 4 independent equations,

$$G_{tt} = T_{tt}, \quad G_{tr} = T_{tr}, \quad G_{rr} = T_{rr}, \qquad (2.6-8)$$

and

$$G_{\theta\theta} = T_{\theta\theta}.$$

Because of the Bianchi identities, satisfied identically by the Einstein tensor and by virtue of (2.3) by the energy momentum tensor, the last equation is a consequence of Eq.s (2.6-8) and can be ignored.

Summary of equations

Einstein's equations,

$$\begin{aligned}
G_t^t &= -e^{-\lambda}\Big(\frac{-\lambda'}{r} + \frac{1}{r^2}\Big) + \frac{1}{r^2} = 8\pi\Big(\mathrm{e}^{-\nu}\rho\dot\psi^2 - \mathcal{L}\Big),\\
G_r{}^r &= -\mathrm{e}^{-\lambda}\Big(\frac{\nu'}{r} + \frac{1}{r^2}\Big) + \frac{1}{r^2} = 8\pi\Big(-e^{-\lambda}\rho(\psi')^2 - \mathcal{L}\Big),\\
G_t^r &= \mathrm{e}^{-\lambda}\frac{\dot\lambda}{r} = -8\pi\mathrm{e}^{-\lambda}\rho\psi'\dot\psi.
\end{aligned} \qquad (2.9-11)$$

Wave equations,

$$\begin{aligned}
\frac{1}{2}\Big(\mathrm{e}^{-\nu}\dot\psi^2 - \mathrm{e}^{-\lambda}(\psi')^2 - 1\Big) &= \frac{dV}{d\rho},\\
\partial_t(\mathrm{e}^{(-\nu+\lambda)/2}r^2\rho\dot\psi) - (\mathrm{e}^{(\nu-\lambda)/2}r^2\rho\psi')' &= 0.
\end{aligned} \qquad (2.12-13)$$

The function λ is often replaced by the function M defined by

$$M := \frac{r}{2}(1 - \mathrm{e}^{-\lambda}), \quad \mathrm{e}^{-\lambda} = 1 - \frac{2M}{r}; \qquad (2.14)$$

then Eq.s (2.9-10) can be written as follows,

$$\begin{aligned}
M' &= 4\pi\, r^2(\mathrm{e}^{-\nu}\rho\dot\psi^2 - \mathcal{L}),\\
r\mathrm{e}^{-\lambda}\nu' &= 1 - \mathrm{e}^{-\lambda} + 8\pi\, r^2(\mathrm{e}^{-\lambda}\rho\psi'^2 + \mathcal{L}).
\end{aligned} \qquad (2.15-16)$$

The two equations can be combined to yield

$$(\nu+\lambda)' = 8\pi\, r\, e^{\lambda}\rho\, (e^{-\nu}\dot\psi^2 + e^{-\lambda}\psi'^2). \tag{2.17}$$

Remark. Comparing our equations with those that appear in the literature, from Tolman [T] onwards, we see that what is normally called "energy density" is identified with $e^{-\nu}\dot\psi^2\rho - \mathcal{L}$. A very important aspect of equations (2.9-13) is the appearance of the function ν. The equations of the traditional approach contain the derivative of this function but not the function itself; it is therefore meaningless to impose asymptotic boundary conditions. Eddington [Ed], and modern authors as well, fix the boundary of a star at the first zero of density and pressure. There is no matching to an outside or asymptotic Schwarzschild metric, and the "mass" of the star is instead defined by the value of the function M introduced in (2.14). This is a major shortcoming of the traditional approach to relativistic stellar dynamics. The traditional pressure corresponds to $\mathcal{L} + e^{-\lambda}\rho\psi'^2$. In the static case $\psi' = 0$ and there is full agreement since, as already pointed out, our $\mathcal{L}$ is interpreted as pressure in the nonrelativistic limit. But here too, a quantity normally interpreted entirely in terms of matter is found generally to depend on the metric as well.

3. The static limit

In the context of constructing an 'interior solution', to be joined to an exterior Schwarzschild metric, it is common to define "static" by fixing the four-vector field, setting the space components to zero and normalizing U_t as in (2.4), thus $g^{tt}U_t^2 = 1$. We define the static limit by

$$\dot\psi = 1, \quad \dot\rho = \dot p = \dot\nu = \dot\lambda = 0,$$

and Eq.(2.11) then does not allow any reasonable alternative to setting $\psi' = 0$. Conversely, if we admit that 'static' implies no flow, $\psi' = 0$, then $\dot\psi$ is independent of position and then the equations demand that it be a constant. Taking $\dot\psi = 1$, in agreement with the nonrelativistic limit, and the only value possible in Minkowski space, is convenient. The full set of equations in the static case is thus as follows,

$$M' = 4\pi\, r^2(e^{-\nu}\rho - p), \qquad (\lambda+\nu)' = 8\pi\, r\, e^{\lambda-\nu}\rho, \qquad \frac{1}{2}(e^{-\nu}-1) = \frac{dV}{d\rho}, \tag{3.1 − 3}$$

with

$$p := \mathcal{L} = \rho\frac{dV}{d\rho} - V.$$

The first two equations agree with the textbooks if we identify $e^{-\nu}\rho - p$ with the energy density (denoted $\hat\rho$). Derivation of the third equation with respect to r gives $(e^{-\nu})'/2 = p'/\rho$, which is known as the hydrostatic equation ([Ed], page 79); this equation is a consequence of the (contracted) Bianchi identities and is used by Chandrasekhar and by Oppenheimer and Volkoff as well [OV]. Conversely, Eq. (3.3) and the Bianchi identities makes the other wave equation (the conservation law) redundant, so that the only thing that is new in our treatment of the static case is the fixing of an integration constant.

Since this point is important we insist on it. Eq.(3.3) is equivalent to the pair

$$(1) \quad \frac{1}{2}(\mathrm{e}^{-\nu})' = (\frac{dV}{d\rho})',$$
$$(2) \quad \frac{1}{2}(\mathrm{e}^{-\nu} - 1) = \frac{dV}{d\rho} \text{ at a point.}$$

The quantity $(\frac{dV}{d\rho})'$ is easily seen to be the same as p'/ρ, and the first equation is then recognized as the hydrostatic condition. There remains the second statement that, as we said, fixes an integration constant. This equation is new, there is no precedent in the traditional theory. We shall see that it is a positive development.

An important corollary is that matching the solution to an exterior Schwarzschild metric constitutes a complete set of boundary conditions, which is not true in the traditional setting. Another application follows.

In the case of an polytropic equation of state Eq.(3.3) takes the form

$$\frac{p}{\rho} = \frac{\phi}{n+1},$$

where ϕ is the gravitational potential (defined by $\mathrm{e}^{\nu} = 1 - 2\phi$) and n is the polytropic index. Defining the temperature by the gas law, $pv = RT/\mu$, where μ is the atomic weight, we convert this to

$$T = \frac{\mu}{R}\frac{\phi}{n+1}.$$

The standard approach yields the same result, but modulo an additive constant. (Compare [L], p. 268-271.) For a polytrope with $n = 3,\ \mu = .5$, with a sharp boundary where pressure and density drop abruptly to zero, and with the ratio mass/radius equal to that of the Sun, this formula yields a temperature at the surface,

$$T = \frac{1}{8}\frac{1}{8.3 \times 10^7}\frac{(2 \times 10^{33})(6.6 \times 10^{-8})}{7 \times 10^{10}} = 2.7 \times 10^7 \text{ degrees Kelvin.}$$

Empty space

We assume that empty space can be viewed as a limiting case of equilibrium. Though it is difficult to interpret the velocity of flow when no matter is present, we set $\psi' = 0$. If it should happen that $dV/d\rho = 0$ (in empty space), then we get the result that the gravitational potential $-\nu/2$ vanishes. Is this a reasonable conclusion?

Let us remember that the idea of freezing the value of $g_{\mu\nu}U^\mu U^\nu$ at unity has its origin in the equation for geodesic motion of a test particle; it is in that case a constant of the motion that can be fixed. So doing we also eliminate an extra, uninterpretable variable, U^t. (It is the relativistic version of the condition that $E - H = 0$.) We are thus talking about a test particle in the gravitational field, with 3-velocity equal to zero (since we are still dealing with equilibrium). But this is absurd, except in Minkowski space. Indeed, if matter is at rest ($\psi' = 0$) in any other gravitational field then there must be a force present, to balance the gravitational pull on the particle; then the motion is not along a geodesic and then we do not expect that $g_{\mu\nu}U^\mu U^\nu$ should be a constant of the motion. That is exactly what Eq.(3.3) tells us. The quantity $dV/d\rho$ represents the balancing force and it is zero only when the gravitational potential is zero. The aptness of the equation is indicated by the fact that taking the derivative with respect to r one obtains the hydrostatic condition.

Thus we regard empty space as a limit of a space filled with matter. This will turn out to be a clue to understanding the Schwarzschild solution.

Equation of state

An equation of state, for isentropic processes, is a relation between density and pressure. This brings the difficult question of what is the correct definition of the density field. In the limit of weak fields there is no doubt that $\sqrt{-g}\,\rho - p$ is the density of mass, whose integral determines the mass parameter in the asymptotic Schwarzschild metric. But it is not clear that the density of mass is what should appear in the equation of state. It appears that there is no general answer, and no clear consensus on this topic. We have already quoted, in the introduction, an instance where an equation of state was used with different choices of density. Being unable to decide this question, one can only say that the equation of state selected may be appropriate in some cases. As a practical matter, it turns out that different interpretations of the "density" in the polytropic equation of state makes very little difference in the cases that we have studied. This also renders moot the question of whether the traditional normalization of the velocity four-vector leads to the correct definition of the density to be used in the equation of state; nevertheless, there is an important matter of principle.

Our point of view is a little different. We are going to take a very simple choice of the functional V, thus defining the theory. Then we let the theory speak for itself. The interpretation of a theory is in its consequences. In fact it will give us plenty of scope for a thermodynamical interpretation.

In this paper we shall study the consequences of taking

$$V = a\rho^\gamma, \quad \text{thus} \quad p = a(\gamma-1)\rho^\gamma, \quad a, \gamma \text{ constants}, \quad \gamma > 1. \tag{3.4}$$

Asymptotic behaviour

Let us consider a matter distribution that extends to great distances and suppose that, as r tends to infinity, $\rho = br^{-3-\epsilon}$ with b, ϵ constant and $\epsilon > 0$ to make the density integrable at infinity. Thus

$$\rho \sim r^{-3-\epsilon} \quad \Rightarrow \quad \begin{matrix} M' \sim (r\lambda)' \sim r^{-1-\epsilon} \\ \lambda' + \nu' \sim r^{-2-\epsilon} \\ \nu \sim dV/d\rho \sim r^{(3+\epsilon)(1-\gamma)} \end{matrix} \quad , \qquad \epsilon > 0. \tag{3.5}$$

For the time being we shall consider only the case when this equation of state applies for all r larger some limit. We are thus dealing with a stellar atmosphere, though there may be nothing else, as in the case of the young giants.

Inspecting (3.5), we are at first tempted to conclude that λ and ν must fall off faster than $1/r$, which would exclude solutions that behave like the Schwarzschild metric at large distances. In this case $1+\epsilon = (3+\epsilon)(\gamma-1)$, or $\gamma = (2\epsilon+4)/(\epsilon+3) > 4/3$. The way to circumvent this is to assume that $\lambda \sim r^{-1}$, for in that case the leading term in λ makes no contribution to $(r\lambda)'$. This term also has to make no contribution to $\lambda+\nu$, so we can assert that there is a constant m such that

$$\lambda \sim \frac{2m}{r}, \quad \nu \sim -\frac{2m}{r}, \quad (m = \text{constant}).$$

This implies that $(3+\epsilon)(\gamma-1) = 1$, or

$$\gamma = 1 + \frac{1}{3+\epsilon}, \quad 1 < \gamma < \frac{4}{3},$$

which is acceptable; the upper limit is slightly less than the value for an ideal, diatomic gas.

Result for polytropic stars without boundary

(1) If the density of the stellar atmosphere falls off fast enough to be integrable at infinity, then at large distances the metric approaches the Schwarzschild metric.

(2) The static boundary conditions, in the case of the isentropic equation of state, specifies the leading terms for large r as

$$\rho \approx b r^{\frac{1}{1-\gamma}}, \quad p \approx a(1-\gamma) b^\gamma r^{\frac{\gamma}{1-\gamma}}.$$

For the metric, if $0 < \epsilon < 1$, the three leading terms for large r are

$$g_{tt} = e^\nu \approx 1 - \frac{2m}{r} + \frac{8\pi b}{\epsilon(1+\epsilon)} r^{-1-\epsilon}, \quad g_{rr} = \mathrm{e}^\lambda \approx 1 + \frac{2m}{r} - \frac{8\pi b}{\epsilon} r^{-1-\epsilon}.$$

(3) The constant γ has to lie in the interval $1 < \gamma < 4/3$.

The assumed constancy of γ is of course an oversimplification of the situation in real stars, see [M] , [Fo] or [KW] page 175.

The Schwarzschild metric

We ask if there is a singular limiting matter distribution in which the metric becomes exactly Schwarzschild. Assume that

$$\mathrm{e}^\nu = 1 - \frac{2m}{r}, \quad m = \text{constant},$$

and then Eq. (3.3) gives us (precisely)

$$\frac{m}{r-2m} = a\gamma\rho^{\gamma-1}, \quad \rho = b(r-2m)^{\frac{1}{1-\gamma}}, \quad b = (\frac{m}{a\gamma})^{\frac{1}{\gamma-1}},$$

which makes sense for $r > 2m$. Eq.s (3.1-2) now read

$$M' = 4\pi\, b\, r^2\big(r + \frac{1-\gamma}{\gamma}m\big)(r-2m)^{\frac{\gamma}{1-\gamma}},$$
$$(\nu+\lambda)' = 8\pi\, b\, r e^{\lambda-\nu}(r-2m)^{\frac{1}{1-\gamma}}.$$

Both expressions must tend to zero in empty space, in the limit that we are looking for, which requires that γ tend to 1 and that $a \geq m$. In the limiting case when $a \approx m$, and γ is very close to 1,

$$M' = 4\pi\, b\, r^3(r-2m)^{\frac{1}{1-\gamma}},$$
$$(\nu+\lambda)' = 8\pi\, b\, r e^{\lambda-\nu}(r-2m)^{\frac{\gamma}{1-\gamma}}.$$

the pressure and the density both tend to zero except at $r = 2m$. But $dV/d\rho$ does not tend to zero; the gradient $(dV/d\rho)' = p'/\rho$ represents the force per unit volume needed to resist the pull of gravity even in the limit of vanishing density.

These conclusions are supported by numerical calculations presented in Section 5.

An analogy may serve to support this interpretation of the Schwarzschild metric. Consider Poisson's equation in the case of a source localized at the origin. One solves Laplace's equation in empty space ($r \neq 0$) and finds that the solution is const.$/r$, which is singular at the origin. A fine argument, using Gauss' theorem, shows that there is a non-vanishing source, given by a distribution concentrated at the origin. An alternative would be to consider a sequence of continuous distributions of matter, converging towards a δ-function distribution. In General Relativity one solves Einstein's equation for empty space. The solution is "singular" at the origin and at $r = 2m$. The presence of a source is manifest, but where is it localized? The region inside the Schwarzschild radius is unphysical [D], so to localize the source at the origin begs the question [Fr]. The singularity at $r = 2m$ turns out to be removable by a coordinate transformation and the naive conclusion is that the source is nowhere! (Skeptical, we examined the possibility that the Einstein tensor, evaluated for the Schwarzschild metric, might contain a distribution of the form $\delta(r-2m)$. It does not, but there is more to this question.)

What the model shows is that a different distribution, not a δ-function but nevertheless concentrated to a high degree at the horizon, yields a metric that is close to the exterior Schwarzschild metric except in a region near the Schwarzschild. We suggest that this type of localized distribution may be what is relevant in the context of Einstein's equations. In that case we could affirm that the source of the Schwarzschild metric is localized at the horizon.

By including a dynamical model of the source of the gravitational field, we impose additional constraints on the theory and are led to a solution of the problem of localizing the source of the Schwarzschild metric field in a physically meaningful region. Without matter or boundary conditions we cannot hope to find the correct interpretation.

Consider the case of a spherically symmetric distribution of matter, in the form of a spherical shell centered at the origin, and of negligible thickness. In the newtonian theory of gravitation this gives rise to a gravitational potential outside the shell, but it has no effect on the inside! Therefore, if one adopts the hypothesis that the Schwarzschild metric is due to a concentration of mass at the horizon, then it is absurd to pretend that the metric field inside is obtained by analytic continuation. Much more natural to suppose that the metric is discontinuous at the horizon; for example,

$$
\mathrm{e}^{\nu} = 1 - \theta(r-2m)\frac{2m}{r} = \mathrm{e}^{-\lambda},
$$

where θ is the unit step function, and the non vanishing components of the Einstein tensor are

$$
G^t_t = G^r_r = \frac{1}{2m}\delta(r-2m), \quad G^\theta_\theta = G^\phi_\phi = \frac{1}{2}\delta'(r-2m).
$$

Incidentally, the odds against the mythical traveller penetrating the horizon during his life time needs to be re-evaluated!

4. Weak fields

At low densities the metric functions λ, ν will be small compared to unity, and Eq.s (3.1-3) take the form

$$
M' = 4\pi\,(r\lambda)' = r^2(\rho - p), \quad (\lambda+\nu)' = 8\pi\, r\rho, \quad \nu = -2a\gamma\rho^{\gamma-1}. \tag{4.1}
$$

It follows from the third equation that $p = -\frac{\gamma-1}{2\gamma}\nu\rho$, so that the contribution of the pressure to the first equation must be omitted in the weak field approximation.

In the unphysical case that $\gamma = 2$ the equations become linear,

$$
\rho = -\nu/4a, \quad \lambda = r\nu', \quad \lambda' + \frac{1}{r}\lambda = -2\pi\, r\nu/a.
$$

Eliminating λ we get the radial Schroedinger equation for a free particle with angular momentum zero,

$$
\nu'' + \frac{2}{r}\nu' + \frac{2\pi}{a}\nu = 0,
$$

One solution is

$$
\nu = c\,\frac{1}{r}\sin(kr), \quad k^2 = \frac{4\pi}{a}, \qquad c = \text{constant}.
$$

However, the value 2 for γ, implying $\epsilon = -2$ is not an option for an infinitely extended distribution.

Let $\gamma = 1 + 1/n,\ n > 1$. Continue to ignore the contribution of the pressure. The equations are now, with $f^n = \rho$,

$$
2M' = \lambda + r\lambda' = 8\pi\, r^2 f^n, \quad (\lambda+\nu)' = 8\pi\, r f^n, \quad \nu = -2a\gamma f, \tag{4.2}
$$

We reduce this to the nonlinear Schroedinger equation (Emden's equation)

$$
\begin{aligned}
&f'' + \frac{2}{r}f' + k^2\, f^n,\\
&\nu = -2a\gamma f, \quad \lambda = r\nu',\ k^2 = 4\pi/a\gamma.
\end{aligned} \tag{4.3}
$$

When $n = 5$ there are exact solutions ([Ed] page 89),

$$f = \alpha(1 + \beta^2 r^2)^{-1/2}, \quad \beta^2 = \frac{8\pi}{6a\gamma}\alpha^4.$$

Thus,

$$\rho = \alpha^5(1 + \beta^2 r^2)^{-5/2}, \quad p = \frac{a\alpha^6}{5}(1 + \beta^2 r^2)^{-3}$$
$$-\nu/2 = a\gamma\alpha(1 + \beta^2 r^2)^{-1/2}, \quad \lambda = r\nu'.$$

Asymptotically, $m := \lim(-r\nu/2) = a\gamma\alpha/\beta$ is the gravitational mass. The integrated density is

$$4\pi \int_0^\infty r^2 \rho dr = 4\pi\alpha^5\beta^{-3} \int z^2(1 + z^2)^{-5/2} dz = (8\pi/6)\alpha^5\beta^{-3} \int (1 + z^2)^{-3/2} dz$$

The last integral equals $\int_o^\infty dx/\cosh x = 1$ and finally

$$4\pi \int_0^\infty r^2 \rho(r) dr = a\gamma\alpha/\beta = m,$$

as expected. In this context ρ is the density of mass. Solutions obtained numerically for $n > 5$ are qualitatively similar. A limiting case of infinite n, or $\gamma = 1$ is discussed in [Ed], page 89. The case of $n < 5$ will be taken up in Section 6.

Remark 1. The relation $\nu = -2a\gamma f$ tells us that the potential and the Emden function are of the same order of magnitude. The hydrostatic equation used by Eddington has the same implication. Therefore, if $n \geq 2$, then consistent application of the weak field hypothesis would require us to drop the term $k^2 f^n$. Nevertheless, we shall refer to these equations as the weak field approximation, since they are the equations proposed and studied by Eddington and generally considered to be valid in the case of weak gravitational fields. See [Ed], page 80.

Remark 2. We shall always require our solutions to be asymptotically Schwarzschild. Besides identifying the gravitational mass with the asymptotic limit of $-r\nu(r)$ it requires that $r\nu(r) + r\lambda(r)$ tend to zero. This is a departure from the work of Eddington. In his work the equation $\nu = -2a\gamma f$ (coming from Eq. (2.12)) is replaced by the hydrostatic equation $\nu' = -2a\gamma f'$. Integration introduces an undetermined, additive constant and matching the metric to Schwarzschild becomes less restrictive. Eddington's boundary conditions are $f(0)$ finite and $f'(0) = 0$.

5. Numerical integration for an infinite star

Numerical solutions of the system (3.1-3) that govern static configurations were obtained with the help of Mathematica.

Parameters

A scale transformation, by which the coordinate r is replaced by a new coordinate

$$x = \sqrt{8\pi a^{-n}}\, r$$

allows, without essential loss of generality, to set $a = 1$ in $V[\rho] = a\rho^\gamma$. (Recall that p and ρ are scalar fields, not densities). The constant γ is expressed as

$$\gamma = 1 + \frac{1}{n}.$$

The choice of the value of n is the only freedom that we have explored with respect to the thermodynamic properties of matter, except for boundary conditions. In this section we restrict n to the regime of infinite stars, $n > 5$.

Boundary conditions

It was seen that the density of a star without boundary is integrable at infinity if and only if the metric is asymptotically Schwarzschild. This result reflects the fact that the equations of the model, unlike the traditional ones, are not invariant under a constant shift of the function ν. It allows us to fix a solution in terms of asymptotic boundary conditions, while all previous work has applied conditions at the center of the star, a region about which little is known in advance. (A notable exception is the paper [C].) Thus Eddington supposes that the metric is well behaved at the origin, more precisely that $\nu'(0) = 0$, and concludes that all stars without boundary have infinite mass. ([Ed], page 88.) It turns out that, in the traditional theory, the solutions that are regular at the origin do not have finite mass; one has to choose. (We are under the restriction $n > 5$.)

It will be useful to compare our results with those of the weak field approximation. The equations used by Eddington [Ed] and in all subsequent work are the same as those of our model, except for two modifications.

1. Our equation (2.12),

$$\frac{1}{2}(\mathrm{e}^{-\nu} - 1) = dV/d\rho$$

differs from the usual hydrostatic condition only in the fact that it determines the integration constant, which gives a meaning to the value of the function ν and not just its derivative. This affects only the boundary conditions. It gives us the possibility of fixing a unique solution by the imposition of asymptotic boundary conditions. The usual practice is to integrate outwards from the origin, this implies regularity near the origin and no solution of this type is integrable at infinity, as Eddington remarks. ([Ed, page 88.)

2. To relate one system of equations to the other we have the dictionary $p \mapsto \hat{p}$, $\mathrm{e}^{-\nu}\rho \mapsto \hat{\rho} + \hat{p}$, so that the equation of state, though formally the same in the two theories, does not involve the same variables.

Solutions are completely fixed by the choice of $\lambda(x_0)$ and $\nu(x_0)$ at some point x_0. By trial it is found that

$$\nu(25) = -.1, \quad \lambda(25) = .1$$

generates solutions for ν, λ, ρ and p that fall off uniformly at large x. In the first place, the point chosen is more than 5 times the value of x where bumps appear, so that $(\nu+\lambda)(25)$ is close to the asymptotic value zero. In the second place, we expect to find solutions for which $x\nu(x)$ and $x\lambda(x)$ tend to finite asymptotic values, which forces the value of $\lambda(25)$ to agree with that of $-\nu(25)$ to within one part in a thousand.

The number $-25\nu(25) = 2.5$ is close to the asymptotic value of $-x\nu(x)$ at infinity; that is, to the parameter $2m$ in the asymptotic field $g_{00} = \mathrm{e}^\nu = 1 - 2m/x$. The only free parameter is thus the gravitational mass m. We have not explored a wide range of values of this parameter but some sampling suggests that things do not change qualitatively. Thus Fig.s 1a-d show that a factor of 4, or even a factor of 40, in the value of $2m$ leaves the basic shapes intact.

Results

1. In a first study we varied the parameter n over the set $\{6, 8, 10, 20, 50, 100, 1000\}$. In the case $n = 6$, Mathematica returns reliable answers over the range $10^{-300} \leq x \leq 100\,000$. As n grows the permissible range of x contracts at the upper end, due to strong cancellations in the intermediary stages of the calculation, but it does not fall below 8 000.

The metric is well behaved near the origin, both g^{tt} and g_{rr} approaching zero linearly with x. However, the Emden function is negative in a very small region near the center, as shown in the table.

Results for $n = 6$ are presented in Table 1 and illustrated in Fig.s 1a-f. The most striking aspect is the strong concentration of the density near the Schwarzschild radius. Since this is the quantity that sets the scale one should not be very surprised.

The functions f, ρ and p take positive values, except very close to the center where the Emden function f turns negative. The existence of a hole, with vanishing or negative pressure, needs interpretation, but the suggestion that the highest densities (and temperatures) occur at a distance from the center is, at least, fascinating.

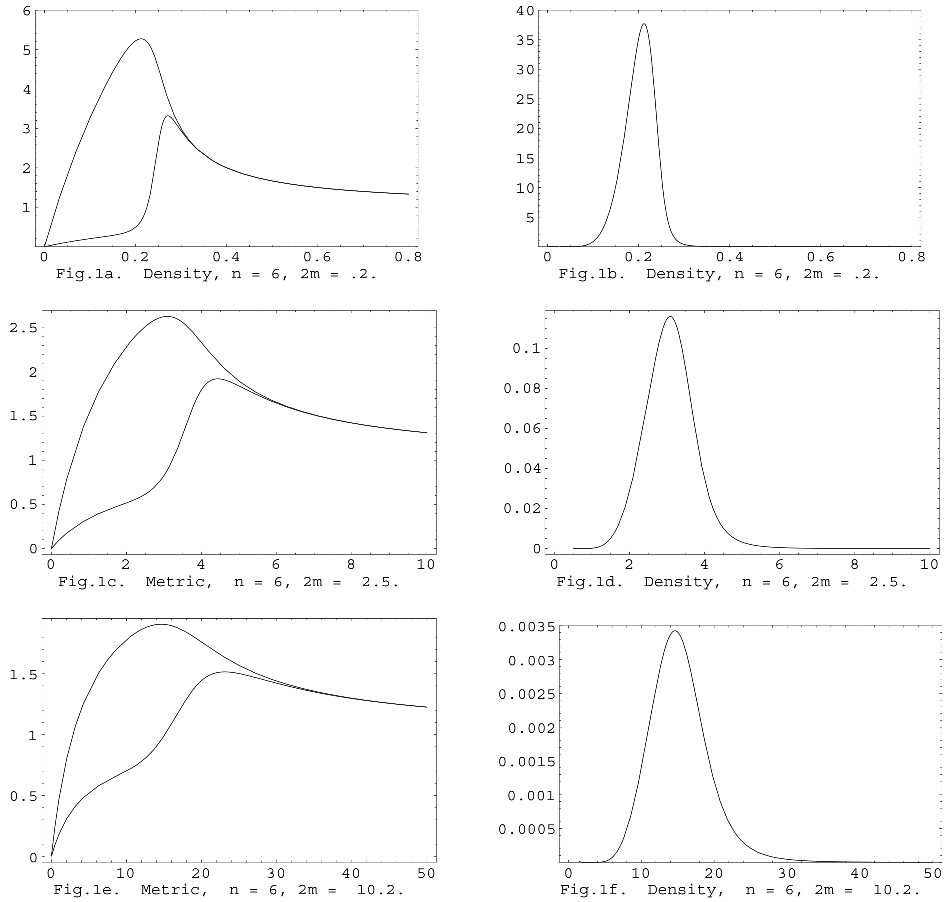

Fig 1. On the left, $e^{-\nu}$ *on top and* e^{λ} *below. On the right, the density. Parameters* $n = 6$, $2m = .2$, *2.5 or 10.2 as indicated.*

.

The functions f, ρ and p take positive values, except very close to the center where the Emden function f turns negative. The existence of a hole, with vanishing or negative pressure, needs interpretation, but the suggestion that the highest densities (and temperatures) occur at a distance from the center is, at least, fascinating.

Table 1, $n = 6$, $2m = 2.38$

x	0.01	1	3	3.2	4.43	10	100	1000	100 000
ν	3.7618	-.43002	-.96572	-.96483	-.75010	-.27165	-.02408	-.002381	-.00002246
λ	-5.269	-1.0731	-.19000	-.02177	.65367	.27148	.02408	.002382	.00002383
$e^{-\nu}$	.02324	1.53729	2.62669	2.62435	2.1172	1.3121	1.0244	1.00238	1.00002
e^{λ}	.0051699	.341964	.826958	.978462	1.9226	1.3119	1.0244	1.00238	1.00002
ρ	.0053810	.0001491	.114806	.113819	.01205	5.73×10^{-6}	1.30×10^{-12}	1.14×10^{-18}	7.94×10^{-31}
p	-.0003754	5.72×10^{-6}	.013340	.013206	.0009616	1.28×10^{-7}	2.26×10^{-15}	1.93×10^{-22}	1.27×10^{-36}
f	-.41861	.23027	.69715	.69615	.47881	.13377	.01044	.001022	9.63×10^{-6}

2. Table 2a shows a comparison of results of the model with the "weak field approximation" (See Remark 1 at the end of Section 4.), with the same boundary conditions and

with the same gravitational mass, $2m = 2.38$, for the case $n = 6$. There is close agreement for $x > 10$, very close for $x > 100$.

Table 2a. Values of $-\nu$, $n = 6$. $2m = 2.38$.

	x	0.01	1	3	3.2	4.43	10	100	1000	100 000
	Weak	-21.36	2.384	.8318	.78006	.563035	.2500	.02500	.002500	.00002402
$-\nu(x)$	Model	-3.762	.4300	.9657	.9648	.750103	.2716	.02408	.002381	.00002246
	Schwarzschild, $-\ln(1-2.38/x)$			1.577	1.362	.7706	.2718	.02401	.002383	.0000238

The inner region is illustrated in Fig.2a-d, where the ordinate is $e^{-\nu}$.

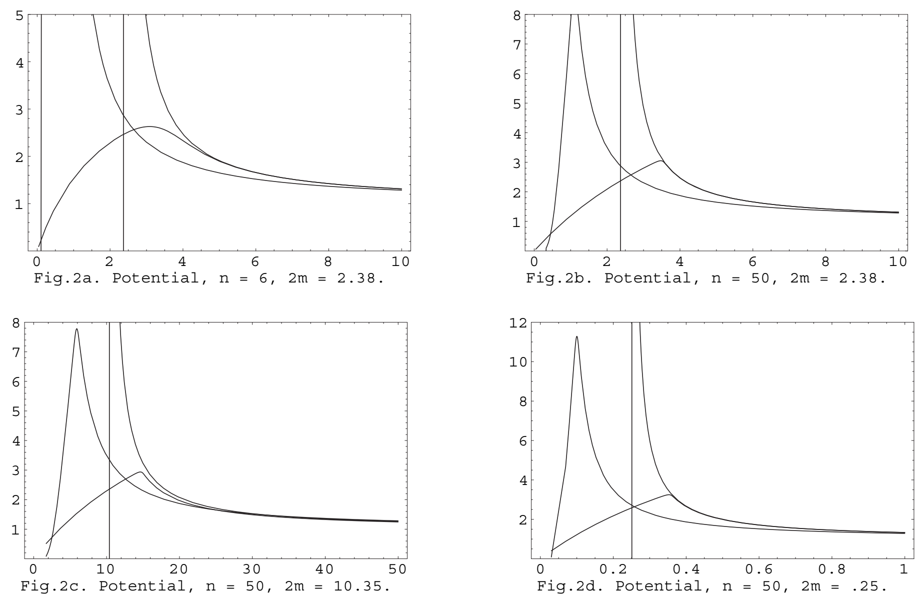

Fig.2. Comparison between the model, the "weak field approximation" and the Schwarzschild metric in 4 cases. The hyperbola is the Schwarzschild metric, $g^{tt} = g_{rr}$. The lower curve is g^{00} in the model. The upper curve, with the sharp peak inside the Schwarzschild radius, is the same function in the "weak field approximation".

Table 2b compares the densities in the model with the densities of the weak field approximation. The remarkable contrast between the model and the weak field approximation must have some importance for our understanding of what is going on near the centers of some stars.

Table 2b. Values of ρ, $n = 6$, 2m = 2.38.

	x	0.1	1	3	3.2	4.43	10	100	1000	100 000
$\rho(x)$	Model	.0013733	.0001491	.1148	.1138	.01205	5.73×10^{-6}	1.30×10^{-12}	1.14×10^{-18}	7.94×10^{-31}
	Weak	1.813	1.137	.002052	.001396	.0001995	1.51×10^{-6}	1.51×10^{-12}	1.51×10^{-18}	1.19×10^{-30}

3. The gravitational mass is given by

$$2m = \lim x\lambda(x) = 2.379, \quad n = 6, ..., 100.$$

This is not the integral of a mass density or an energy density, although there is an expression for this number in terms of an integral

$$2m = 2M(\infty) = \int_0^\infty (\mathrm{e}^{-\nu}\rho - p)x^2 dx + M(0).$$

The numbers for $n = 6$ are $2m = 2.379 = 4.303 + (-1.924)$. The integral is not the integral of a density.

To give a sense of the effect of larger values of n we present the following Table 3, showing the maxima of the function $g^{00} = \mathrm{e}^{-\nu}$, the functions λ and the density field ρ.

Table 3

n:	6	10	20	100
Max(ρ)	.1160	.2353	.5162	2.6526
Max($\mathrm{e}^{-\nu}$)	2.630	2.9036	3.0317	3.040
at x =	3.086	3.2613	3.3811	3.5125
Max(λ)=	.6537	.8326	.9650	1.0717
at x =	4.432	4.012	3.76815	3.6055

The position of the density bump does not change much, but the amplitude shows a strong rise. Since the total gravitational mass remains the same there must be concentration of mass close outside the Schwarzschild radius. This is clearly illustrated by Fig.s 3a-c, where one sees a dramatic narrowing of the bump at about 1.5 times the Schwarzschild radius.

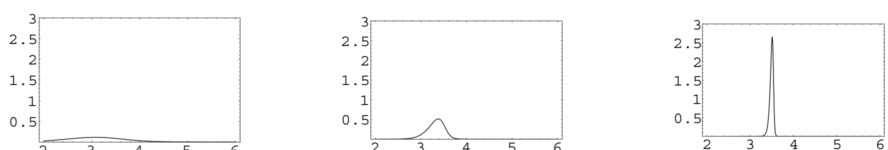

Fig.3a. Density when n = 6. Fig.3b. Density when n = 20. Fig.3c. Density when n = 100.

Fig.3. The density peak narrows as n is increased.

4. Finally, Fig.s 4a-c illustrate the gradual approach of the function $g^{00} = \mathrm{e}^{-\nu}$ towards the Schwarzschild limit as n grows. This function, for large n, has a very sharp bend where it joins up with the Schwarzschild metric. Matter is strongly concentrated at this point and the density is very low inside. (Fig.4d will be explained later.)

As is seen from these last figures, we have not been completely successful in finding a limiting equation of state that reproduces the Schwarzschild metric all the way down to the Schwarzschild radius. The association of such a limit with large polytropic index seems, however, to be confirmed. A closer apprach will be obtained later.

We think that the results support our contention that the Schwarzschild metric should be interpreted in terms of a stiff equation of state and a concentration of matter at $x \leq 2m$.

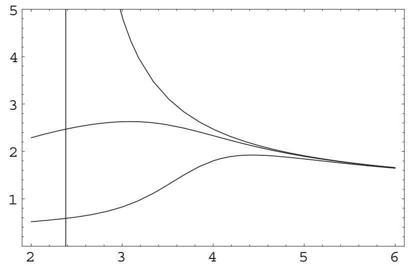

Fig.4a. Model vs Schwarzschild, n = 6.

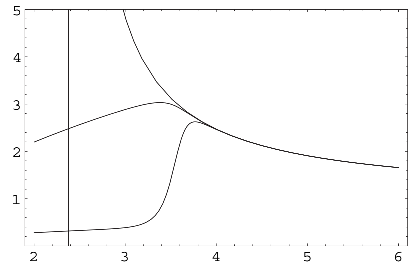

Fig.4b. Model vs Schwarzschild, n = 20.

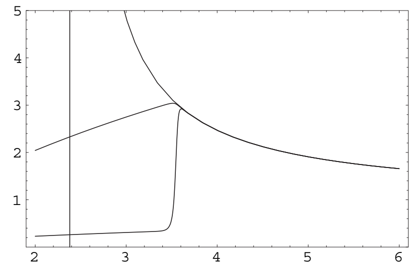

Fig.4c. Model vs Schwarzschild, n = 100.

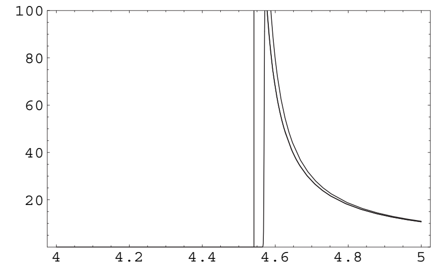

Fig.4d. Metric with density cut-off.

Fig.4. As n is increased g^{tt} and g_{rr} cling closer to the Schwarzschild metric. In the last figure the effect is enhanced by making the radius finite.

Conclusions, infinite stars

The main result of this investigation of infinite stars with finite mass is that they seem to have a chance to exist. The metric is smooth all the way to the center. The low density found in the interior is interesting, although the persistence of a small central region in which the pressure turns negative is disturbing. The most optimistic explanation is that it results from adopting a somewhat naive equation of state, with the same polytropic index throughout. Perhaps the correct attitude to this problem was defined by Chandrasekhar [C]: "It is of course clear that if we agree to describe the outer parts of a configuration by singularity possessing solutions of the differential equations involved, then we must assume that somewhere in the inner regions the perfect gas laws break down."

6. Finite stars

Eddington adapted the polytropic model to stars with boundary. It is useful to return for a moment to the weak field approximation.

Weak fields

For a finite matter distribution the restriction $n > 3$ is no longer relevant. Equation (4.3) was integrated numerically for many values of n by Emden [Em], and some of the results are quoted by Eddington [Ed]. As we pointed out in connection with Eq.s (3.1-3) the last of those equations, namely Eq.(3.3),

$$\frac{1}{2}(\mathrm{e}^{-\nu} - 1) = \frac{dV}{d\rho} \tag{6.1}$$

is not normally invoked in the traditional approach; instead the 'hydrostatic condition'

$$-(\mathrm{e}^{-\nu})'/2 = p'/\rho, \tag{6.2}$$

is used, or the weak field approximation $\nu'/2 = p'/\rho.$. However, here Eddington integrates this equation and chooses the integration constant (which is regarded as insignificant) so as to end up with (6.1) precisely. Consequently, our equations reduce exactly to those used by Eddington in the approximation of weak fields. This approximation was always taken to be justified since the metric fields in most stars (except neutron stars and hypothetical super massive stars) are in fact weak. See, however, Remark 1 at the end of Section 4.

Boundary conditions

In accord with the principle that the gravitational field is determined by the mass that lies closer to the origin, Eddington looked for solutions that are finite at the origin and satisfy $\nu'(0) = 0$. To apply the theory to a finite star, Eddington considered the case $n < 5$, to take advantage of the fact that, in this case, solutions of Eq.(4.3),

$$f'' + \frac{2}{r}f' + k^2 f^n = 0$$

that are finite and positive at the origin pass through zero at a finite value of the radius. Since the pressure turns negative, this point must lie outside, or at the boundary, of the region where the theory is applicable. Eddington supposed that the first zero of the pressure indicates the boundary of the star. This implies that the gravitational potential also vanishes at the surface, and because Eddington's choice ot the constant in his integration of (6.1) brings him into agreement with the wave equation Eq.(3.3), we can identify his gravitational potential with $-\nu/2$. However, as already pointed out in the preceding section, Eddington cannot fix a solution by matching for $g_{00} = \mathrm{e}^\nu$ to a Schwarzschild metric on the outside, which is why he must impose boundary conditions at the origin, thus having to estimate the conditions that are likely to prevail at the center of the star. The 'mass' is defined as the integral of the density. In the nonrelativistic case this definition of the mass agrees with our definition in terms of the asymptotic metric.

An essential feature of our model is that the integration of the hydrostatic equation is determined without ambiguity and matching up to an outside metric is no longer trivial. Outside the star the metric is that of Schwarzschild, and in particular $\nu + \lambda = 0$. In the weak field approximation λ is eliminated by means of the relation $\lambda = r\nu'$, so we must have

$$r\nu'(r) + \nu(r) = 0, \; r \geq R.$$

This implies the boundary condition

$$Rf'(R) + f(R) = 0, \tag{6.3}$$

which is incompatible with $f(R) = 0$.

The equation of state that is being used in the interior is not expected to apply to matter near the surface. The idea that the pressure drop continuously to zero at the boundary is reasonable, but we are not sure that the same can be said of the density. Oppenheimer and Volkoff, who also fix the boundary of the star at the point where the pressure drops to zero, seem to agree with this, for they say that there are many equations of state that allow the pressure to go to zero at finite density. The one they use does not have that property [VO]. In order to compensate to some degree for the shortcomings of the polytropic equation of state we take a positive value for $f(R)$. Since the boundary value of $f'(R)$ is fixed in terms of $f(R)$, we shall get a 1-parameter family of solutions.

This applies to the weak field approximation as well as to the model. In the case that $n = 1$, $\gamma = 2$ Eddington's solution is

$$\nu(x) = -\frac{c}{x}\sin\frac{x}{2}, \quad X = 2\pi, \quad c = \int r^2\rho(x)dx),$$

where $x = X$ is the boundary, corresponding to $r = R$. Our solution, in the weak field approximation (valid in this case) and with the boundary conditions imposed by the model, is

$$\nu(x) = -\frac{2m}{x}\sin\frac{x}{2}, \quad X = \pi,$$

where m is the gravitational mass.

This result is significant. In this particular case, when $n = 1$ and in the weak field approximation, the boundary condition (6.3), that comes from the matching of the interior metric to the exterior Schwarzschild metric, unexpectedly selects the only solution that is regular at the origin, thus vindicating, in this case at least, some of Eddington's precepts.

It was found, empirically, that this last observation applies to the exact solutions of the full model as well. Namely, if the parameters R and $\nu(R)$ are not chosen with special care, then the solution has a very singular behaviour at the origin. This suggests that Eddington is right in choosing regular boundary conditions at the origin; this also guarantees that pressure and density both remain positive throughout. The least that can be said is that the stars that satisfy this criterion form a distinguished family, characterized by a critical relationship between radius and gravitational mass. We shall see that this implied relationship between radius and mass is not one-to-one; for fixed values of n and R there are from 1 to 3 possible values of the mass.

Numerical results for finite stars

Fig.s 5a-d show examples of the contrast between Eddingtons star, where density and pressure both tend continuously to zero at the boundary, and the solution obtained from the model, matched to Schwarzschild at the boundary, where density and pressure both drop abruptly to zero. These are the boundary conditions that we impose from now on.

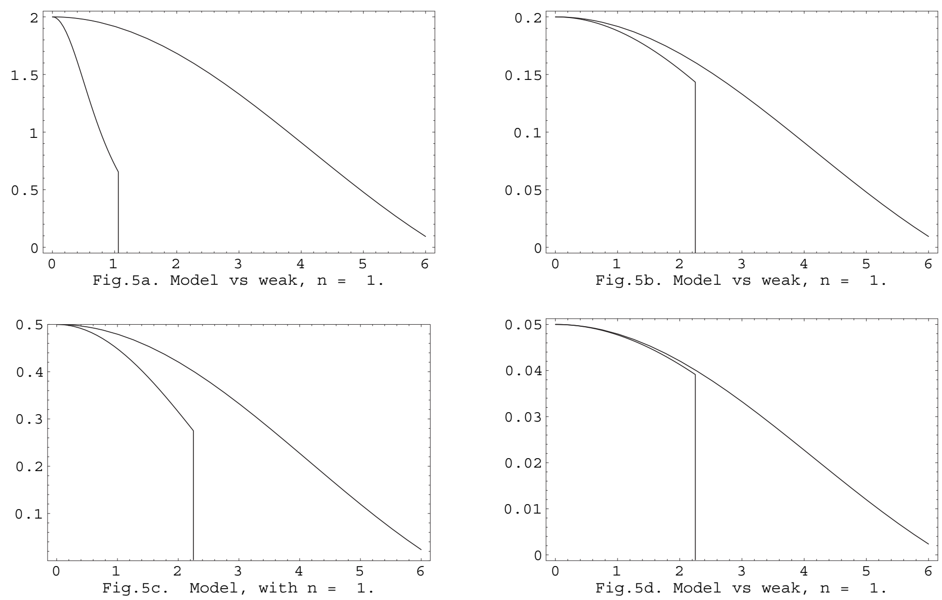

Fig.5. The pressure, for different central pressures. Comparison between Eddington's star, with boundary at $x = 6$ and the result of the model, which puts the boundary at the end of the short curve, where the metric is matched to the external Schwarzschild metric.

There is a critical, empirial relationship between the radius R and the gravitational mass, both being determined by the conditions of matching the Schwarzschild metric at $r = R$,

$$\nu(R) = -\lambda(R) = \ln(1 - \frac{2m}{R}),$$

that leads to a metric that is regular at the origin. It is found that most solutions (of the exact equations of the model, as in the weak field approximation) blow up at the origin, and that the exceptional case when ν remains bounded can be localized to a very high degree of accuracy by the condition that $\nu'(0)$ vanish. Thus to find these particular solutions we adopt Eddington's boundary condition $\nu'(0) = 0$. Starting the integration at the origin (actually at $x = 10^{-10}$), varying the value of $\nu(0)$ from .001 up to the maximum value that the computer program allows, we integrate up to the first zero of $\nu + \lambda$, where the star ends, to obtain a parametric representation of the critical relation between R and $2m$. The results are presented in Fig.6a-d as a log-log plot and juxtaposed in Fig.7.

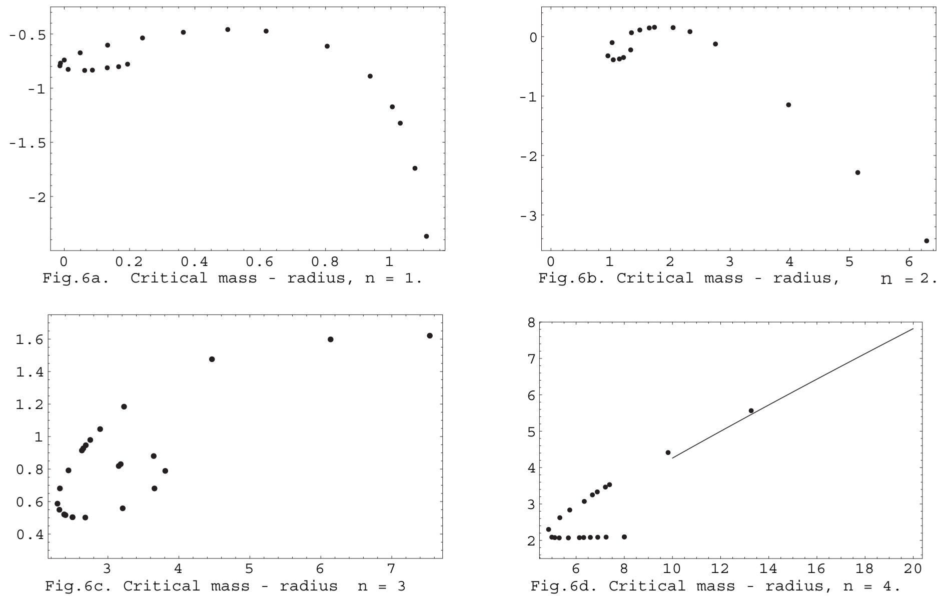

Fig.6a-d. The relation between mass and radius in a log-log plot. The abscissa is the radius and the ordinate is 2m.

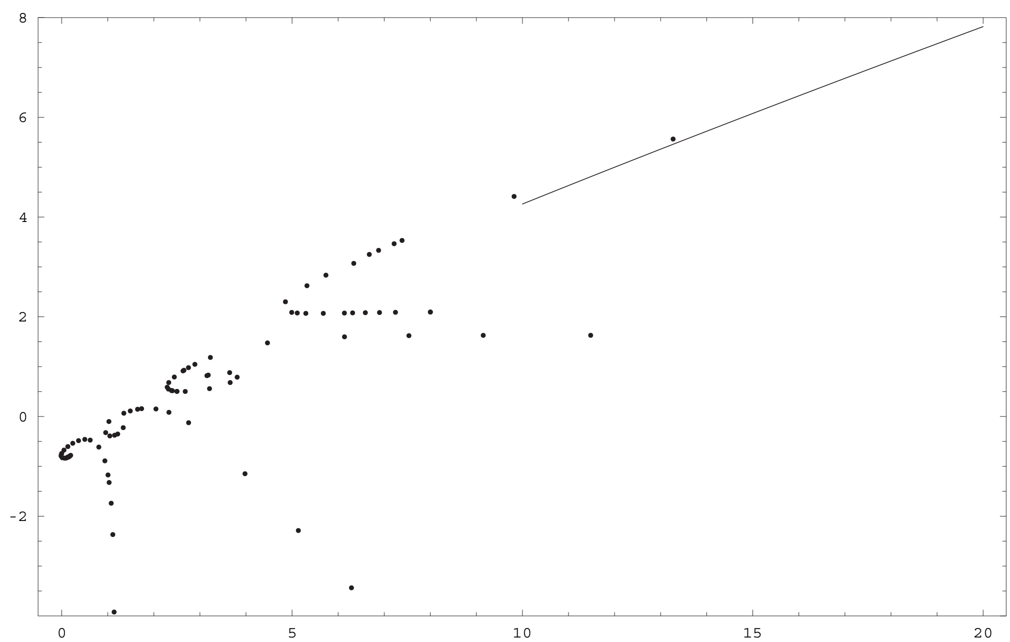

Fig. 7. Critical mass - radius, n = 1- 4

Fig.7. A composite of Fig.6a-d.

For $n = 4$, in the upper part,$-\nu(0) \geq .4385$, the relation of X to m is well represented by $\ln(2m) = .567\ (\ln X)^{.876}$. In the figures it is represented by this curve, to distinguish this branch from the other, in the region where they nearly coincide.

The strange shape of the curves gives rise to several different configurations with the same gravitational mass but different size. Table 4 gives an example for $n = 3$.

Table 4.

$2m$	2.49	2.41	2.21
X	38.3	24.0	9.85
$-\nu(0)$	1.5	2.0	.7

There are stars intermediary between the first two, both of which are unstable, suggesting aging with small mass loss, but none between the last two, which indicates a cataclysmic event. The star with the smallest radius is stable. The number $-\nu(0)$ is a measure of the temperature at the center.

Neutron stars and Oppenheimer-Volkoff theory

In their ground breaking paper [OV] on neutron cores, Oppenheimer and Volkoff used a very stiff equation of state. In order to to compare the predictions of their theory with our model we shall have to do so with the same equation of state and choose for this purpose the polytrope with $n = 1/10$. Numerical results are presented in Table 5.

Table 5. $n = 1/10$.

	Model			Oppenheimer-Volkoff [OV]		
-$\nu(0)$	.001	.01	.097	.001	.01	.097
X	.07439	.20855	.5512			
-$\nu(X)$	.0006637	.00613	.062	0.00067	.00656	.06329
$2m$	.00004936	.0013746	.0331	.000049	.00137	.03376
$\rho(0)$	.3679	.4634	.5842	.3679	.4634	.5842
$\rho(X)$	.3532	.4445	.5576	.3532	.4448	.5588

The most significant parameter is $2m/X$, the potential at the surface relative to the maximum possible that is reached when the surface of the star coincides with the Schwarzschild horizon. The highest value found in our model for this equation of state ($n = 1/10$) is only .06. This explains the near coincidence of the results; the gravitational fields are weak.

The big contrast between the two theories is that the model imposes stronger conditions for matching up with the external Schwarzschild metric, to wit, the condition $\nu(X) + \lambda(X) = 0$. According to Oppenheimer and Volkoff the boundary of the star is not at the point $x = X$, and indeed X is not defined in their theory, but at the first zero of the density. (Our computer program did not allow us to carry the integration that far.) The values $\nu(X), ..., \rho(X)$ on the right side were read at the corresponding values of X computed from the model.

The question of the greatest possible mass for a neutron star is too complex to be raised here; see [H]. We venture only to propose that the upper limit on the parameter $2m/X$ is about .06 for small n, but values up to 2/3 are possible with polytropic index $n = 1$.

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