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Sources for gravity
The Noetherian field theories

Christian Fronsdal

Dept. Physics and Astronomy, University of California los Angeles, USA

ABSTRACT. This work began with the intention to clear up some uncertainty about the definition of the Cauchy-Noether energy-momentum tensor of relativistic and non relativistic field theories, the natural source of Einstein's equation. The result is a relativistic fluid that is described by an action principle and includes vortex motion, and a non relativistic limit with applications to aerodynamic lift, capillarity, metastable configurations and superfluids.

The paper opens with a brief review of the development of an idea first advanced by Maupertui (1698-1759): the dynamical Action Principle. The story reached a climax with the creation of Einstein's theory of General Relativity in 1915 and the work of Noether that accompanied that event. Her work served as an inspiration for particle physics for 100 years. The discovery of gravity waves (LIGO 2016) showed that Gravity is a phenomenon akin to, and part of, particle physics, to be treated as a canonical, Noetherian field theory and, eventually, quantized. We determine a class of Noetherian field theories that provide sources for Gravitation and that have sensible and useful non relativistic limits.

Highlights include: 1. The crucial contribution of the notoph, the 2-form gauge theory of Ogievetskij and Polubarinov. 2. The multiple roles that are played by 'permittivities'. 3. Planetary rings are a natural phenomenon. 4. Future applications to Galactic velocity curves and Dark Matter.

I. Introduction

First of all, to justify a review, we shall argue that there are challenging difficulties, in each of three closely intertwined areas of theoretical physics.

Hydrodynamics

The most important tool in the theory of flows is the Navier - Stokes equation. It is based on an invention by Cauchy, the stress - energy tensor. The main idea is that this tensor is divergence-less, a property related to conservation laws. But supplemental equations have always been needed, including an expression for the energy. Unfortunately, we do not have any guidance for choosing this expression even in the simplest cases. As the system under study gets more complicated the internal inconsistencies become more serious; while at the same time the degree of arbitrariness grows and we lose all predictive power.

Thermodynamics

A beautiful theory was devoloped during the 19'th century, applicable to one-component homogeneous fluids at equilibrium. But so far there is no uniform approach to binaries and more complicated systems, or to systems away from equilibrium. The common approach is to investigate one phenomenon at a time, applying a different method for each, without a concern for overall consistency. This leaves too much freedom and the result is, again, poor predictive power.

Gravitation

General Relativity was created between 1905 and 1915. The milestone that was reached at the end of that period was the discovery of an action for the metric field. Unlike Hilbert and some others, Einstein was not looking for an action principle but for a set of differential equations. But the equations are very complicated, while the action is very simple, so this became the accepted way to look at it, at first.

The result was a beautiful theory for the metric, applicable to the case that no matter is present (Einstein 1917). It is based on an action principle with the action

$$A[g] = \int d^4x\sqrt{-g}R[g],$$

where $R[g]$ is the Riemannian curvature scalar. Variation of the action with respect to the metric gives

$$\delta A[g] = \int d^4x\delta(\sqrt{-g}R) = \int d^4x\sqrt{-g}\ \delta g_{\mu\nu}G^{\mu\nu}$$

- this defines the Einstein tensor G - and the field equation

$$G_{\mu\nu} = 0.$$

General Relativity becomes a general theory of gravitation when we know the source ($T_{\mu\nu}$) in the equation

$$G_{\mu\nu} = T_{\mu\nu}.$$

This is where Noether comes in. Following the path that her work suggested leads to the source of gravity and to substantial progress in 3 major areas.

Much important work has been done in all of these fields and our intention is to continue that work. What we suggest is that we take particle physics as a model, and ask: to what does that discipline owe its **amazing power of prediction**? Is there a chance of emulating its success in other fields? We hope to show that the theory of gravitation is one where the approach has been too easy going, where fundamental principles have not been implemented. We insist that the interaction of the metric with matter fields must be a Noetherian field theory and we show that this emphasis has a strong effect in the other fields, most dramatic in Hydrodynamics where we discover a Noetherian field theory that includes flows with vorticity and predicts internal stresses in fluids.

Progress in the theory of gravity has an important and immediate application in the non relativistic limit that is hydrodynamics. Conversely, a formulation of hydrodynamics as an action principle points to a relativistic theory of gravitation.

Hilbert was concerned about symmetries and conservation laws, and with an action that was invariant under all coordinate transformations, expecting that conservation laws must be the key. Einstein, at first unaware of the connection, did not expect conservation to be important, but he reached the same point of view a little later. Hilbert turned the problem over to Emmy Noether, an algebraist. She was familiar with the energy momentum tensor of Cauchy, or she became aware of it during her work on the metric, and she made a most capital discovery (Noether 1918):

The natural context for a conserved energy momentum tensor is a field theory, defined by an action constructed from scalar fields and a Riemannian metric.[1] It is based on a scalar action density

[1] Not just scalar field; some generalizations are well known and more will be carefully examined later.

(Lagrangian density $\mathcal{L}$) constructed exclusively from fields $\phi, \psi, ...$, their first order derivatives $\phi_\mu, \psi_\mu, ...$ with respect to the space time coordinates and the components of the metric.

At this point in the history of physics leading scientists turned their creative genius to the development of quantum theory, while the development of the general theory of gravitation ground to a complete halt. This is clearly documented by Misner, Thorne and Wheeler (1972) in their influential textbook. We shall review this important development in the next section.

Quantum Theory and Particle physics

Let us digress for a moment to review the meteoric advance of what is today referred to as Particle Physics.

One feature that characterized quantum theory from the beginning was that it correlates experimental facts with amazing efficiency: a single parameter and Ohm's law, in the context of Bohr's atomic model, were enough to give a detailed, numerical account of atomic spectra. Heisenberg and Pauli created the first quantum field theory. Dirac added a beautiful mathematical structure and the result was that quantum theory became a very tight, axiomatic. This led to the demand for internal mathematical consistency and, eventually, to an understanding of the Lamb shift and to ultraviolet renormalization. And here we may call attention to the fact that, what enabled them - Heisenberg and Pauli - to do this was a return to Hamiltonian mechanics. Faddeev and Popov (1967) took the same path to create perturbative quantum gravity. The associated Lagrangian and action became ever more prominent and essential for the progress that eventually led to the electro-weak theory, and the Standard Model.

The point that we wish to make is that **the amazing power of prediction of quantum theories is strongly correlated to the fact that they were formulated as action principles.** Equally important was the rigid attention to mathematical concistency.[2] It is evident that to ignore 'theoretical' or aesthetic requirements is to loose predictive power.

During all of this development of quantum theory there was an acute awareness of the importance of conservation laws and of the debt that was due to Noether. There are two aspects to her work. For Einstein and Hilbert

[2] In this case: renormalizability.

it was invariance under the infinite dimensional group of general coordinate transformations that helped them understand their theory. Particle physics focused on the finite dimensional subgroup of transformations that leave the metric invariant, and it was this application of Noether's work that was important in quantum theory, while invariance under general coordinate transformations played a lesser role. But this is what concerns us here, as we end the digression to particle physics.

II. A brief history of action principles

The first to formulate a principle of maxima or minima may have been Bernoulli. The mathematical formulation was developed by Euler and Lagrange and led to what we call Lagrangian or Hamiltonian mechanics. An application to minimal surfaces is attributed to Plateau. (See Fomenko, 1989.) On the way a most important contribution was made by Maupertui (1698-1759) who invented the Action and who was the first to formulate a dynamical variational principle, applicable to systems in motion.

According to the textbooks that were in use, 50 - 60 years ago, by students in their first years of physics, the work of Lagrange was motivated as follows. He was aware of the practical importance of changing coordinates, and impressed by the amount of labor that this involved. This led him to look for a reformulation of Newton's equation that would involve quantities with simpler transformation properties. The Lagrangian solves this problem admirably, for it is a scalar with respect to coordinates and only the Lagrangian needs to be transformed. It contains velocities but no accelerations, and it defines the dynamics. Today the standard way to deal with the Coriolis force begins by transforming the Lagrangian to moving coordinates.

But there was another advantage. As Poincaré wrote:

"*We cannot content ourselves with formulas simply juxtaposed which agree only by a happy chance; it is necessary that these formulas come as it were to interpenetrate one another. The mind will not be satisfied until it believes itself to grasp the reason of this agreement, to the point of having the illusion that it could have foreseen this.*" (Poicaré 1908)

In Lagrange's formulation of mechanics we are not confronted with a set of equations that agree by happy chance, but by a single function and a concise statement to the effect that actual motions minimize the action.

From this compact statement all the dynamical equations are derived. Also important: the general experience is that the set of equations derived from an action principle have a good chance of being internally consistent.

The paradigm for Lagrangian mechanics is a theory with general variables $q_1, ..., q_n$ and a Lagrange function depending on the q's and on the time derivatives $\dot{q}_1, ..., \dot{q}_n$. Here we are exclusively interested in Lagrangian field theories, the first on record being the following.

The first Action for Hydrodynamics

The variables of classical hydrodynamics are a scalar field ρ, the density, and a vector field $\vec{v}$, a velocity. The fundamental equations are the equation of continuity

$$\dot{\rho} + \vec{\bigtriangledown} \cdot (\rho\vec{v}) = 0 \tag{2.1}$$

and an equation attributed to Bernoulli, Euler and Lagrange. To Lagrange alone is given the honor of having united the two equations in an action principle, with the action (Lagrange 1760)

$$A = \int dt d^3x \mathcal{L}, \quad \mathcal{L} := \rho(\dot{\Phi} - \vec{\bigtriangledown}\Phi^2/2 - \varphi) - W[\rho]. \tag{2.2}$$

In this theory the velocity field is restricted to the form of a gradient,

$$\vec{v} := -\vec{\bigtriangledown}\Phi. \tag{2.3}$$

There is one pair of canonical field variables, the density ρ and the velocity potential Φ. Gravity is represented by Newton's gravitational potential φ; at this point an external field. The two Euler-Lagrange equations are the equation of continuity (from variation of Φ) and the Bernoulli equation (from variation of ρ),

$$\dot{\Phi} - \vec{\bigtriangledown}\Phi^2/2 - \frac{\delta W[\rho]}{\delta\rho} = 0. \tag{2.4}$$

The two equations (2.1) and (2.4) are the core of classical and modern hydrodynamics. The action principle is evoked in some textbooks, as in Lamb (1932) and Fetter and Walecka (1980).

The strong appeal of this theory is evident in the development of the theory of aerodynamic lift. The limitations implied by Eq. (2.3) are vividly illustrated in Birkhof's review of wind tunnels (Birkhoff 1950). It soon became apparent that the limitation (2.3) excludes any possibility of lift and drag. Considerable success was achieved when this condition was applied only locally, but there was no theory based on a variational principle. And that, we think, is the challenge that should be our guidance.

Thermodynamics

These early developments of hydrodynamics were followed by 100 years of decisive progress in thermodynamics. By mid-century Massieu (1876) reached the conclusion that a thermodynamic system (at equilibrium) is completely defined by any one of a set of fundamental potentials, each of which is qualified to be called energy. This statement was underwritten by Gibbs and its general validity has not been challenged. By this time the energy concept was deeply entrenched and the idea that the energy defines the system was becoming implicit in treatises on thermodynamics. Some of the greatest minds of the 19'th century made sustained attempts to formulate thermodynamics as an action principle; among them Helmholtz was the most persistent. Poincaré was another physicist who hoped to find an action for thermodynamics. Helmholtz work of 1883 was critically reviewed by Poincaré (1908) who devoted a whole chapter of his own book to it.

Gibbs paper of 1878 made minimum energy and maximum entropy the cornerstones of the theory. Maxwell was greatly inspired by Gibbs and was chasing an action principle when he died prematurely in 1879. See Ruckeyser 1942. But neither Gibbs, nor any of his numerous followers, exploited the action principle that was lying just under the surface of this work.

It is difficult to understand the lack of perseverance of these attempts of the 19'th century, for both Gibbs and Helmholtz were very close. A part of the answer is in the tentative action for a homogeneous system at rest,

$$A(S,P,V,T) = F(V,T) + ST + VP.$$

Here F is the free energy, supposedly given; the entropy S and the pressure P are fixed parameters and the manifold of physical configurations is defined by the Euler-Lagrange equations

$$\frac{\partial F}{\partial T} + S = 0, \qquad \frac{\partial F}{\partial V} + P = 0.$$

The emergence of special relativity in 1905 captured most of the attention until we come to the pivotal discoveries of 1915-1918, Einstein's - and Hilbert's - theory of 1915, the great challenge of finding the source for the dynamical metric, and Noether's treatise of 1918.

III. Einstein, Hilbert and Noether

Einstein alone is credited with the creation of General Relativity, but Hilbert independently discovered the action

$$A[g] = \int d^4x\sqrt{-g}R$$

only 5 days later. The two men competed, but they collaborated as well, and Hilbert's influence may have been decisive; it was Hilbert who recruited Emmy Noether to assist in unraveling the mystery of symmetry and conservation. Noether, on the other hand, must surely have been influenced by the work of Cauchy (1789-1857), although his stress tensor was completely phenomenological.

What concerns us here is Noether's second theorem and invariance under general coordinate transformations. The great contribution of Noether was the discovery of the natural environment for the energy momentum tensor.

The second theorem (Noether 1918). Consider a Riemannian space with an arbitrary, symmetric connection; that is, the metric g and the connection Γ are, in principle, unrelated. On this space consider a theory of scalar and tensor fields defined by an action

$$A_{\rm matter}[g_{\mu\nu}, \phi, \psi, ..., \phi_\mu, \psi_\mu, ...] = \int d^4x\sqrt{-g}\mathcal{L}, \tag{3.1}$$

invariant under general transformations of the coordinates.

Eq.(3.1) implies that the Lagrangian density $\mathcal{L}$ is independent of the connection, a severe restriction that rules out higher spins, about which more later. See Misner, Thorne and Wheeler (1972). That allows the connection to be the metric connection, so that the covariant derivatives of the metric tensor are zero; that is the connection that will be used from now on.

Suppose further that the fields $\phi, \psi, ...$ satisfy the Euler-Lagrange equations of the action $A_{\rm matter}$. Then the tensor field with components

$$\hat T_\mu^\nu := \sum_\phi \phi_\mu \frac{\partial\mathcal{L}}{\partial\phi_\mu} - \delta_\mu^\nu\mathcal{L} \tag{3.2}$$

satisfies the covariant divergence condition

$$(\hat T^{\mu\nu})_{;\nu} = 0. \tag{3.3}$$

The proof is an easy calculation in the case that the metric coefficients are independent of the coordinates; it amounts to calculating the integrals $\int d^4x\sqrt{-g}\partial_\mu\mathcal{L}$ under the stated conditions. When the components of the metric do depend on the coordinates, we use the fact that the connection is the natural one: that the covariant derivatives of the metric tensor vanish.[3] Then the calculation goes through by interpreting the derivatives of the matter fields as covariant derivatives. The covariant derivative acts as a derivation of the tensor algebra and allows to carry through the partial integrations as well, as usual.

The lack of uniqueness of the conserved tensor defined by (3.3) alone is notorius. It is a general experience that this ambiguity is of no physical significance and that there is one that is symmetric.

We omit details of the proof but verify the statement in the special case of the relativistic action associated with Lagrange's theory, when

$$A_1 = \int d^4x\sqrt{-g}\left(\frac{\rho}{2}(g^{\mu\nu}\psi_\mu\psi_\nu - c^2) - f - sT\right). \tag{3.4}$$

The action (2.2) is the non relativistic limit of (3.4), with (Fronsdal 2007a)

$$\psi = c^2t + \Phi, \quad g_{00} = c^2 - 2\varphi.$$

In this case the Noetherian energy momentum tensor is

$$\hat{T}_{\mu\nu} = \rho\psi_\mu\psi_\nu - g_{\mu\nu}\mathcal{L}. \tag{3.5}$$

We call attention to the factor ρ, interpreted as a mass density. It plays the same role as the factor ϵ, the susceptibility, in electrodynamics and it is an essential dynamical variable in hydrodynamics. In empty space ρ is a constant and the theory reduces to one of the field theories of Particle Physics

The significance of these result will become clear in the next section.

[3] See Misner, Thorne and Wheeler (1972).

IV. The field equations

Variations of the metric action,

$$A[g] = \int d^4x\sqrt{-g}R \tag{4.1}$$

with respect to the metric,

$$\delta A]g] = \int d^4x\sqrt{-g}\,\delta g^{\mu\nu}G_{\mu\nu}, \tag{4.2}$$

defines the Einstein tensor

$$G_{\mu\nu} = \frac{\delta R}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}R.$$

and gives a unique field equation

$$G_{\mu\nu} = 0. \quad \textit{Einstein's equation} \text{ in vacuo} \tag{4.3}$$

It is a set of second order differential equations for the components of the metric, an elaborate generalization of Poisson's equation for the Newtonian potential. It makes no reference to any other fields or sources; it characterizes the metric field in a space time that is empty (a 'vacuum') except for the metric itself.

The discovery of field equations for the gravitational metric <u>in empty space</u> was a milestone in the development of a Theory of Gravitation. It has received a direct confirmation only recently, with the discovery of traveling gravitational waves (LIGO 2016). But it is not yet a theory of gravitation. Just as the mass density ρ appears as a source for the potential in Newton's theory, we need to add sources to the right hand side of (4.3). This hydrodynamical source has remained undecided for 100 years.

We shall take great pains to justify this last statement. The difficulty is related to the Bianchi identity.

The Bianchi identity

Theorem. The Einstein tensor, defined by (4.2), satisfies the following equation,

$$D_\nu G_\mu{}^\nu = 0, \quad \textit{Bianchi identity} \qquad (4.4))$$

identically. The operator D_ν is the covariant derivative, with the metric connection.

Proof. The action (4.1) is invariant under infinitesimal coordinate transformations,

$$\delta x^\mu = \xi^\mu,$$

under which

$$\delta g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu.$$

Hence

$$\delta A[g] = 2\int d^4x\sqrt{-g}(\mathcal{D}_\mu \xi_\nu)G^{\mu\nu} \tag{4.5}$$

is identically zero. An integration by parts gives

$$\int d^4x\sqrt{-g}` \xi_\nu D_\mu G^{\mu\nu} = 0.$$

Since the vector field ξ is arbitrary this validates the statement (4.4).

Implications of the Bianchi identity

Let us add a source to Einstein's equation (4.3):

$$G_{\mu\nu} = T_{\mu\nu}. \qquad \mathit{Einstein's\ Equation}\ \text{cum fontibus} \tag{4.6}$$

Taking the covariant divergence we find

$$D_\nu G^{\mu\nu} = D_\nu T^{\mu\nu} = 0. \tag{4.7}$$

The first expression is identically zero; therefore an inevitable consequence is that **there can be no solution of Einstein's equation for any source unless**, by virtue of the field equations of the matter fields,

$$D_\mu T^{\mu\nu} = 0. \qquad \mathit{The\ Bianchi\ constraint} \tag{4.8}$$

Eq(4.8) is a condition of integrability. At first, this condition was treated with due respect and led to the first examples of interactive gravitation theory, among the most important a theory of the metric tensor field interacting with the vector field of electromagnetism, Einstein-Maxwell theory. This is where Emmy Noether entered history.

We have seen that, if a matter action of the type (3.1) is constructed from the metric and from matter fields only, then there is a tensor field

$$\hat{T}^\nu_\mu = \sum_\phi \phi_\mu \frac{\partial \mathcal{L}}{\partial \phi_\nu} - \delta^\nu_\mu \mathcal{L} \tag{4.9}$$

that satisfies the 'conservation law' (3.3) on shell. The first idea is to identify $T_{\mu\nu}$ with $\hat{T}_{\mu\nu}$, this being a solution of the Bianchi constraint, and the first example bears this out. In the case that the matter Lagrangian is (3.4)

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} A_{\rm matter} \tag{4.10}$$

coincides with the Noether tensor (3.5). But if we try to extend this success to invent a coupling between the metric and electromagnetism then we find that in that case $\hat{T}_{\mu\nu}$ is not symmetric, while (4.10) clearly is. Misner, Thorne and Wheeler (1972) turned away from electromagnetism at this point but it is well known that the conserved tensor of Maxwell's theory can be modified to make it both symmetric and conserved.

Therefore, let us explore the action

$$\int d^4x\sqrt{-g}\mathcal{R} + kA_{matter}, \quad k = \text{const.},$$

with A_{matter} as in (3.1). The field equation is

$$\frac{-1}{k} G^{\mu\nu} = T^{\mu\nu} := \frac{\delta}{\delta g_{\mu\nu}} A_{\rm matter}.$$

This tensor is not proportional to the Noether tensor, but it is evidently symmetric, and **it is conserved**.

Proof. Calculate the total variation of the matter Lagrangian under an infinitesimal coordinate transformation $\delta x^\mu = \xi^\mu$. On shell; that is, by virtue of the matter field equations it is,

$$\delta A_{matter} = \int d^4x\sqrt{-g}\Big(\delta g_{\mu\nu} T^{\mu\nu} - \xi_\mu D_\nu \hat{T}^{\mu\nu}\Big).$$

This quantity is zero by invariance; the second term is zero by Noether's theorem; therefore the first term on the right is also zero.

For example, in the case of the electromagnetic field,

$$\mathcal{L} = F^2, \quad T_{\mu\nu} = 2g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{2} F^2 g_{\mu\nu},$$

and, on shell

$$(T_\mu{}^\nu)_{;\nu} = F^{\alpha\beta}(dF)_{\beta\mu} = 0.$$

Conclusion

We have found that a class of relativistic field theories, the **Noetherian field theories** defined in (3.1), provide sources for Einstein's field equation. It would be satisfying to be able to state that this is the only way. This we cannot do, as the following is intended to show.

Following the lead of Misner, Thorne and Wheeler, we invoke the Palatini formalism. In a Riemannian space with metric g and curvature Γ let $R[g,\Gamma]$ be the well known expression for the curvature scalar. Take

$$A[g,\Gamma] = \int d^4x\sqrt{-g}R[g,\Gamma].$$

and let

$$A_{matter}[g,\Gamma,\phi,\psi,...]$$

be the action of an ivariant field theory. If Γ is not the metric connection, then the Bianchi identity is not satisfied, for a coordinate transformation will affect both g and Γ. There will, of course, be a more complicated condition of integrability, but we have learned that it is nothing more that an expression of invariance. That is, the only condition of integrability is that the action must be invariant.

This is perhaps disappointing, but we can fall back on the fact that there are other reasons why the metric connections is preferred, including the importance of geodesics. We shall probably not be making a mistake if we conclude that the Noetherian field theories must be the first choice for the source of Einstein's equations. And this, finally, is what gives the theory its power of prediction.

V. The simplest sources

The action (3.4) is the simplest one possible. We can generalize it by adding the energy momentum tensors of any number of similar, spin-less matter field theories. What about field theories with spin?

The proof of Noether's theorem depends on the fact that the matter action involves matter fields and the metric only, which is true of (3.4). This condition is also satisfied by Maxwell's theory and any relativistic field theory in whch the covariant derivatives can be replaced by ordinary derivatives, as in vector gauge theories. Finally it is true in the gauge theory of the relativistic 2-form that was studied by Ogievetskij and Polubarinov (1964).

If we restrict our attention to ordinary hydrodynamics, with the usual 4 degrees of freedom, then the simplest possibilities are:

Type 1. The action for the spinless field is $A_1[\rho,\Phi]$ in (3.4). The non relativistic limit is the Lagrange action (2.2) of 1760. It can be mixed with electromagnetism by using the conserved current, to give it electric charge.

Type 2. A 2-form field $Y = (Y_{\mu\nu})$ with the Lagrangian density dY^2 that was analyzed by Ogievetskij and Palubarinov (1964). It mixes with electromagnetism in

$$A_2[Y,F] = \int d^4x\Big(\sqrt{-g}(\rho dY^2 + \epsilon F^2) + \gamma dY dF\Big). \tag{5.1}$$

to give a mass to the photon. It does not have a condition of continuity and it has no conserved mass flow. This field theory, as well as its non relativistic limit, was used by Lund and Regge in their work on vortices on superfield Helium.

Type 3. Two or more Type 1 theories can be combined to describe hydrodynamic or thermodynamic mixtures, without vorticity.

Type 4. Much more interesting is a combination of Type 1 and Type 2, including a mixing term,

$$A_4[\rho,\Phi,Y] = A_1[\rho.\Phi] + A_2[\rho,Y] + \kappa\int d^4x\rho dY d\psi. \tag{5.2}$$

The non relativistic hydrodynamic limit in a physical gauge is

$$A_4[\rho,\Phi,Y] = \int d^4x\Big(\rho(\dot\Phi + \dot{\vec X}^2/2 + \kappa\dot{\vec X}\cdot\vec\bigtriangledown\Phi - \vec\bigtriangledown\Phi^2/2 - \varphi) - W[\rho]\Big). \tag{5.3}$$

Here the electromagnetic fields have been omitted. The coupling term (the κ-term) is included, for it is the source of vorticity. The vector field $\vec X^i = \epsilon^{ijk}Y_{jk}$; the components Y_{0i} are zero in the physical gauge. This is a model of compressible hydrodynamics with an equation of continuity, vorticity (if $\kappa \neq 0$) and internal stress; it is the only realistic model of hydrodynamics with the canonical 4 degrees of freedom. This theory has already been applied to several interesting problems, including Couette flow, menisci and fluid stress (Fronsdal 2015, 2017, 218a,b, 2019a,b). The relativistic version can be used for planets, for rotating heavy bodies and Black Holes, and for the dynamics of The Milky Way to explain the anomalous velocities.

Apropos the scalar field ρ. Its inclusion in A_1 is clearly essential to obtain the correct non relativistic limit. It is an essential novelty that was missing in early attempts to use particle field theories to construct sources for Einstein's equation. It is analogous to the classical treatment of electromagnetic fields in an extended medium. A version of electromagnetism has a dynamical permitivity (Fronsdal 2007b). It may have an application to the theory of ectromagnetic fields on a background of a photon gas; or it may be used as an infrared regularizing device. Even gravity can be modified by including a density factor in the action. Infrared regularization is a prerequisite for quantization of Gravity (Weinberg 1970), (Marciano and Sirlin 1975).

VI. Concluding remarks

Gravitational waves have been found to travel through empty space; like photons, they do not need a medium. The fields of the first category (Type 1) are apparently different; they propagate by oscillations of one or more densities; they are the fields associated with diverse fields of matter that we know.

To keep the overall picture as simple as possible it is tempting to suppose that the 2-form that is needed for vorticity is a fundamental field like gravity and electromagnetism; that there is only one, universally coupled to matter. Clearly it will make a contribution to the total mass and energy of the Universe; this makes it a natural candidate for playing the role of Dark Matter. It interacts with matter, it manifests itself in phenomena like liquid stress that are difficult to recognize. It may make the photon locally massive and it makes a significant contribution to gravitational lensing. It will affect the orbits of stars around galaxies. I short, it is a prime candidate for Dark Matter.

Another type of hydrodynamics that seems to be very special is superfluid Helium. There is a widespread feeling that Helium is very much like a common fluid that behaves in unaccustomed ways because it remains fluid at very low temperatures. If we consider a normal gas, with a boiling point of $2.19K$, with 2 liquid, miscible phases , and promote it to type 3 by including the field $\vec{X}$, we shall have a fluid with only 6 degrees of freedom. If we try to apply this theory to Helium, adjusting the equation of state and the single free parameter κ, then we shall be in a position to make very precise predictions about all aspects of the fluid, right or wrong. Incidentally, we would attempt to replace the suggestion (London 1938) that the superfluid component have zero entropy by the conjecture that it have very low adiabatitic index - less than 1/10.

An accompanying paper applies the theory (Type 3) to the structure of rotating planets.

Wikidedia recalls the controvercy that was ignited by Maupertuis' last paper (1744), concerning the several kinds of contributions of velocity to the action.

We read: "Cartesian and Newtonian physicists argued that in their collisions, point masses conserved both momentum and relative velocity. (...) Leibnizians, on the other hand, argued that they also conserved what was called live force or vis viva. (....even at rest...) there is an inherent velocity in bodies; when they begin to move, there is a second velocity term corresponding to their actual motion. This was anathema to Cartesians and Newtonians. An inherent tendency towards motion was an occult quality of the kind favoured by mediaeval scholastics and to be resisted at all costs." This was the beginning of the escalating conflict that cost Maupertui his job, the transfer of Euler from Berlin to St. Petersburg; Lagrange was called to replace him in Berlin.

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