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A classical mistake and what it tells us.

How to do better with an action principle for Hydro and Thermo dynamics

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ABSTRACT Rayleigh’s stability analysis of cylindrical Couette flow, of 1889 and 1916, is in contradiction with observation. The analysis is repeated in many textbooks and reviews up to 2017, and its failure to agree with observation was duly noted, more successful approaches have been found, but little was done to discover the weak point of Rayleigh’s argument. This paper identifies the mistake as one that is endemic in the literature. Since the physics of the problem remains poorly understood, a discussion should prove useful. Briefly, the argument depends on the Navier-Stokes equation and on the assumption that a certain expression called “energy density” or “kinetic potential” can be interpreted and used as such. It is shown that the use of any expression for a kinetic potential is in conflict with the Navier-Stokes equation, in all but a very limited context. An alternative analysis of basic Couette flow, based on an **Action Principle** for compressible fluids, provides a Hamiltonian density as well as a kinetic potential. The two are not the same, even in the simplest cases. With the kinetic potential provided by the action principle, Rayleigh’s stability criterion is a necessary condition for stability, but the action principle applies to compressible flows and a new stability criterion achieves a better agreement with observation. The action is Galilei invariant. Several new experiments are suggested.

I. Introduction

An action principle gives us an efficient and elegant formulation of hydrodynamics (Fetter and Walecka 1980). (The action principle is actually due to Lagrange (1760). The action in modern notation is

$$A_{FW} = \int dt \int d^3x \left(\rho(\dot{\Phi} - \vec{\nabla}\Phi^2/2 - \phi - W[\rho]) \right). \quad (1.1)$$

The velocity field is irrotational,

$$\vec{v} = -\vec{\nabla}\Phi, \quad (1.2)$$

which is a strong limitation. There have been many attempts to lift it to a more general context, especially in dealing with superfluids, and some have come very close to formulating a more general action principle. The strong hint provided in a widely acclaimed paper (Hall and Vinen 1956) was overlooked.

Traditional hydrodynamics has 4 independent variables, the density and the 3 components of the velocity. The action (1.1) has only 2 independent variables; two more are needed. But it will not do to seek a direct generalization of Eq.(1.2).

This is because the scalar velocity potential is essential, for several reasons. In the first place, the Euler-Lagrange equation

$$\frac{\delta}{\delta\Phi}A_{FW} = \dot{\rho} + \vec{\nabla} \cdot (\rho\vec{v}) = 0 \quad (1.3)$$

is the equation of continuity, the very essence of hydrodynamics.

As important is the tradition that incorporates gravitation into field theories, especially into hydrodynamics, by including the Newtonian potential in the Hamiltonian density. (Granted that it is suggested by the correspondence between hydrodynamics and particle mechanics, but our subject is field theories.) Hydrodynamics is a non-relativistic approximation to General Relativity, but is there a generally-relativistic field theory that has the expected non-relativistic limit? Yes, there is.

Consider the generally relativistic field theory with action (Fronsdal 2007)

$$\int d^4x \sqrt{-g} \left(\frac{\rho}{2} (g^{\mu\nu} \psi_{,\mu} \psi_{,\nu} - c^2) - W[\rho] \right). \quad (1.4)$$

To explore the non-relativistic limit set

$$g_{00} = c^2 - 2\varphi, \quad g_{11} = g_{22} = g_{33} = -1,$$

other components zero, expand (1.4) in powers of $(1/c)$ and take the limit $1/c \rightarrow 0$. To ensure the cancellation of the terms of lowest order we need to set

$$\psi = c^2 t + \Phi,$$

then terms of order c^2 cancel and we are left with the action (1.1). In the context of field theories this is the best (perhaps the only) confirmation of the expectation that Non-Relativistic Hydrodynamics, with the inclusion of the gravitational potential φ , is the

non-relativistic limit of General Relativity. Without the expansion $\psi = c^2 t + \Phi$ the role of gravitation in hydrodynamics would lose the only direct contact that it has with General Relativity. Equation (1.3) leads to the correct transformation law for the field Φ under Galilei transformations, given that ψ is a scalar field on space time. But this theory is not enough, because it is limited to irrotational flows.

The equations of hydrodynamics are Galilei invariant, but the Hamiltonian is not; only the action is invariant; the search for a Galilei invariant expression for the energy is mis-directed. See *e.g.* Khalatnikov (1955 page 57), Putterman (1974 page 22).

We conclude that the required generalization of (1.1) must include the field Φ and additional fields with 2 degrees of freedom, with an action that reduces to (1.1) by projection. Two additional dynamical variables are needed, one additional pair of canonical variables, either an additional scalar field or else a gauge field with only one propagating component.

The simple introduction of another scalar velocity potential would not accommodate rotational flow, and the only appropriate relativistic gauge theory is the theory of the massless 2-form field ($Y_{\mu\nu}$) of Ogievetskij and Polubarinov (1964), the theory of the notoph.

The connection of potential flow to an action principle is intimately related to the fact that the velocity is a gradient of a more basic field, a scalar potential. The velocity field of an alternative, ‘Lagrangian’ formulation of hydrodynamics, usually denoted \vec{x} , is also a derivative and also a natural candidate for a canonical variable. We shall call this vector potential \vec{X} , to distinguish it from the coordinates. Since the equations of motion are second order in the time derivative this field appears to have 6 independent degrees of freedom. We already have 2 degrees of freedom with ρ and Φ , conventional hydrodynamics has 4, so this would be far too much. It turns out, however, that it is possible to impose a constraint that reduces the number of degrees of freedom of the new field to 2. The constraint leads to the non-relativistic limit of the relativistic notoph gauge theory, with a single propagating mode. The complete Lagrangian density is, in a fixed gauge,

$$\mathcal{L} = \rho(\dot{\Phi} + \dot{\vec{X}}^2/2 + \kappa\rho\dot{\vec{X}} \cdot \vec{\nabla}\Phi - \vec{\Phi}^2/2 - \phi) - W[\rho]. \quad (1.5)$$

The aptness of this Lagrangian will be confirmed in the next section, by comparison with Navier-Stokes theory.

The full, relativistic gauge theory is well known (Ogievetskij and Palubarinov 1964, Fronsdal 2014).

Constraint and Euler-Lagrange equations

Constraints are typical of gauge theories and arise from variation of the action with respect to a gauge field, in this case the constraint is

$$\vec{\nabla} \wedge \vec{m} = 0, \quad \vec{m} := \rho\vec{w}, \quad \vec{w} := \dot{\vec{X}} + \kappa\vec{\nabla}\Phi. \quad (1.6)$$

This is the constraint that reduces the number of additional degrees of freedom carried by the field \vec{X} to 2. The gauge is fixed by setting the gauge field to zero and this field does not appear in our gauge-fixed, non-relativistic Lagrangian. The Euler-Lagrange equations are as follows.

Variation of Φ gives the equation of continuity,

$$\dot{\rho} + \vec{\nabla} \cdot \rho \vec{v}, \quad \vec{v} := \kappa \dot{\vec{X}} - \vec{\nabla} \Phi. \quad (1.7)$$

The flow vector is thus $\rho \vec{v}$, with \vec{v} defined in (1.7). Variation of \vec{X} leads to

$$\frac{d}{dt} \vec{m} = 0. \quad (1.8)$$

The field \vec{m} that appears in (1.6) and in (1.8) has been called ‘the momentum’ (Lund and Regge 1976), it is distinct from $\rho \vec{v}$. Finally, variation of the density gives, when the force of gravity is neglected, the integrated Bernoulli equation.

$$\dot{\Phi} + \dot{\vec{X}}^2 / 2 + \kappa \rho \dot{\vec{X}} \cdot \vec{\nabla} \Phi - \vec{\nabla} \Phi^2 / 2 = \mu. \quad (1.9)$$

The significance of the parameter κ will become clear as we study the applications. The action of the Galilei group on the notoph field is a gauge transformation; the gauge-fixed field \vec{X} is inert.

The proposed Lagrangian, Eq.(1.5), has much in common with modern theories of rotational flow; they all involve potential flow plus additional velocity fields (Fetter 2009). From Eq.s(1.1-4) it is seen that the vorticity field is

$$\vec{\nabla} \wedge \vec{v} = \kappa \vec{\nabla} \wedge \vec{w} = \kappa \vec{\nabla} \frac{1}{\rho} \wedge \vec{m} = \left(\vec{\nabla} \frac{\kappa}{\rho} \right) \wedge \vec{w}; \quad (1.10)$$

it is orthogonal to the density gradient:

$$\vec{\nabla} \rho \cdot (\vec{\nabla} \wedge \vec{v}) = 0. \quad (1.11)$$

Incompressible fluids can be understood only as a limit within a theory of compressible fluids. Supporting this statement is the fact that there can be no incompressible fluids in a relativistic theory.

II. Cylindrical Couette flow

Couette flow has been studied for 130 years, with the focus on the spontaneous onset of turbulence. The experiment has a homogeneous fluid contained in the space between two concentric cylinders, as in Fig.(2.1), both cylinders turning independently. A completely satisfactory treatment has yet to be found (Lin 1955, Betchov and Criminale 1967, Joseph 1976, Drazin and Ried 1981, Schmid and Henningson 2000).

A pioneering series of experiments, by Couette (1887-1890) and Mallock (1888) was followed by an important paper by Taylor (1923). This work included new experiments as well as a penetrating analysis, the main impact of which was the vindication of the ‘non-slip’ boundary condition. It is the statement that the limit of the material flow velocity, at the boundary of the containing vessel, is the same as the local velocity of the container

at the same point. (Rayleigh, writing in 1916, was already using the non-slip boundary condition without comment, using it in the proof of the theorem under discussion.)

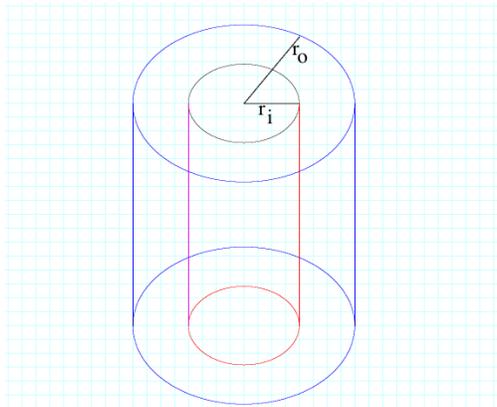


Fig.2.1. Cylindrical Couette flow, both cylinders turning independently.

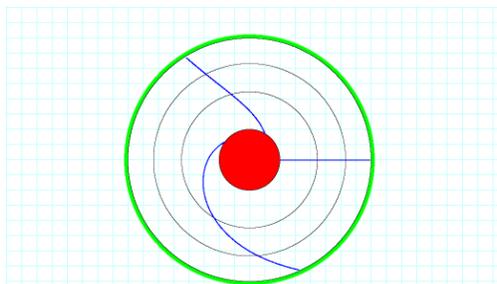


Fig.2.2. Illustration of Cylindrical Couette flow. The curved lines are successive snapshots of a set of particles originally in the radial configuration.

Experiments start with both cylinders at rest, then they are rotated, independently, with angular speeds ω_i, ω_o that are increased slowly, until instability of the flow is observed. The most interesting presentations of experimental results have been in the form of a partition of the plane of angular speeds into a stable and an unstable region. The line, of shape similar to a hyperbola, marks the onset of instability, the stable region lies below. See Fig. 5.2.

Laminar Couette flow

The simplest type of flow velocity is horizontal, stationary and circular, in cylindrical coordinates,

$$\vec{X} = \omega(r)(-y, x, 0), \quad r := \sqrt{x^2 + y^2}. \quad (2.1)$$

The boundary conditions are non-slip. To simplify the analysis one thinks of the cylinders as being long, and ignore end effects.

The particular case

$$\vec{X} = b(-y, x, 0), \quad b \text{ constant,}$$

is the flow of a solid body. It gives rise to a kinetic potential

$$K = -\dot{\vec{X}}^2 \propto -r^2, \quad (2.2)$$

and a centrifugal force $-\vec{\nabla}K \propto \vec{r}$ (all proportionalities with positive coefficients) that is balanced by a pressure (constant $+r^2$) that presents a force $-\vec{\nabla}p \propto -\vec{r}$. It follows, for ordinary fluids with a positive adiabatic derivative $dp/d\rho$, if there are no other forces acting, that $K + p$ is stationary and that the density must increase outwards. See Fig 2.1.

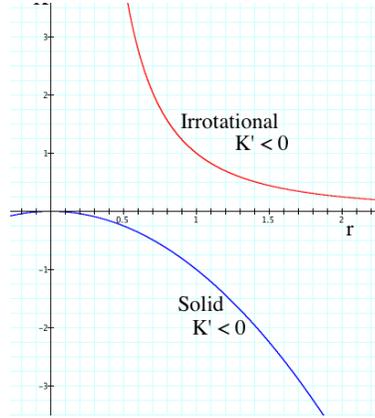


Fig.2.1. The kinetic potentials.

Disturbing question: what is the expression for the ‘kinetic potential’ when the velocity has 2 parts, one rotational and one solid-body type?

The Navier-Stokes equation

is the basis for every traditional analysis of Couette flow:

$$\frac{D}{Dt}\vec{u} = -\frac{1}{\rho}\vec{\nabla}p + \bar{\mu}\rho\Delta\vec{u}. \quad (2.3)$$

Here $\bar{\mu}$ is the kinematical viscosity of the fluid, in the simplest case a constant parameter. This equation agrees with (the gradient of) (1.9) in the special case that $\vec{u} = -\vec{\nabla}\Phi$ and $\dot{\vec{X}} = 0$. It also agrees with (1.9) in the complimentary case, when $\vec{\nabla}\Phi = 0$ and $\vec{u} = \kappa\dot{\vec{X}}$. (The velocity field in (2.3) is denoted \vec{u} to emphasize that it should not be prematurely identified with either $-\vec{\nabla}\Phi$ or $\dot{\vec{X}}$ or \vec{v} .)

Stationary flow is possible only if the effect of viscosity is negligible, this requires that $\bar{\mu} = 0$ or else that

$$\Delta\vec{u} = 0. \quad (2.4)$$

When the flow is stationary, and in the planes perpendicular to the axis, as observed at low speeds, then this condition (with the boundary conditions) allows the general horizontal flow

$$\vec{u} = \frac{a}{r^2}(-y, x, 0) + b(-y, x, 0), \quad a, b \text{ constant}. \quad (2.5)$$

The first is irrotational for $r > 0$,

$$\frac{a}{r^2}(-y, x, 0) = a\vec{\nabla}\theta, \quad \theta = \arctan \frac{y}{x}.$$

This gives rise to a kinematical potential

$$K \propto \vec{\nabla}\Phi^2 \propto 1/r^2 \tag{2.6}$$

and a centrifugal force

$$-\vec{\nabla}K \propto \vec{r}/r^4,$$

and in this case too the force points outwards. Both of these types of flow are of great interest, especially so in connection with superfluid Helium. The second term in (2.5) is the flow of a solid body.

These elementary considerations already suggest that a difficulty is lying in wait for us; compare (2.2) and (2.6). What is the kinetic potential when the velocity field is of neither type, but a combination of both, as in (2.5)?

This prediction of the Navier - Stokes equation is a brilliant success, being in very good agreement with experiments, at low speeds of rotation. The only difficulty is that there is no clear guide to constructing an expression for the energy density, or a kinetic potential. It would be natural to expect that the expression

$$\rho\vec{u}^2/2$$

might be an important part of it, serving as a kinetic potential and giving rise to the centrifugal acceleration; we have seen that this not the case in general, though it is true in the case of irrotational flow, as we shall now show, again.

A question of sign

Consider the simplest cases. The total derivative in (2.3) is

$$\frac{D}{Dt}\vec{u} := \frac{d}{dt}\vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{u}.$$

When the flow is of the solid body type, Eq.(2.2),

$$(\vec{u} \cdot \vec{\nabla})\vec{u} = b^2(-y\partial_x + x\partial_y)(-y, x, 0) = -b^2(x, y, 0) = -\vec{\nabla}\vec{u}^2/2.$$

That is, the acceleration is outwards and the centrifugal potential K is $-\vec{u}^2/2$. In the complimentary case of potential flow,

$$(\vec{u} \cdot \vec{\nabla})\vec{u} = \frac{a^2}{r^2}(-y\partial_x + x\partial_y)\frac{1}{r^2}(-y, x, 0) = -\frac{a^2}{r^4}(x, y, 0) = \vec{\nabla}(\vec{u}^2/2);$$

it is in the same direction, hence correct, but now the centrifugal potential is $\vec{u}^2/2$. The difference in sign between the two cases means that

One can not, in general, associate the Navier Stokes equation with an energy $E[\vec{u}]$ or with a kinetic potential $K[\vec{u}]$.

This conclusion is long overdue.

A classical mistake and what it tells us

To theorists the principal aim has been to understand the limits on the stability of the laminar flow that was described above and the first notable attempt to do so was that of Rayleigh (1889, 1916) who concluded that the laminar flow should be stable if and only if

$$\omega_o > \omega_i \quad (\text{approximately}).$$

Here $\omega_o(\omega_i)$ is the angular velocity of the outer (inner) cylinder and ω_i is positive by convention. In particular, this implies that the laminar flow would be unstable whenever the two cylinders rotate in opposite directions. The result of observation was different: the condition for stable Couette flow is, approximately,

$$\omega_o^2 > \omega_i^2,$$

which allows for stable flow for either sign of ω_o/ω_i .

Raleigh's reaches his conclusion twice.

1. Angular momentum (about the axis of rotation) is conserved, the angular momentum is $L = vr$ "... so that the centrifugal force acting on a *given portion* of the fluid is L^2/r^3 ". That is, L^2/r^2 is regarded as a kinetic potential. But this is without justification and, as it happens, actually wrong; if there is any kinetic potential K in the context of the Navier-Stokes equation it must satisfy

$$(\vec{u} \cdot \vec{\nabla})\vec{u} = \vec{\nabla}K. \tag{2.7}$$

2. We quote (Rayleigh 1916): "We may also found our argument upon a direct consideration of the kinetic energy of the motion.....". He concludes that stability requires that this function increase outwards. This is the same argument and here we see more clearly what the mistake is: There is no justifiable way to introduce the energy concept, or to choose an expression to serve as energy, except in the context of an action principle.

Since the Navier - Stokes equation is an expression of the balance of forces one must ask if there is an additional force that must be taken into account. Yes, there is; but what is without justification is the claim that there is an 'energy density' proportional to L^2/r^2 and that it functions as a kinetic potential.

We have seen (1) that any valid expression for 'kinetic potential' must have opposite signs for the two types of flow and that (2) if there is a suitable expression for a kinetic potential in the context of the Navier-Stokes equation, then it is not the energy density.

Rayleigh's choice of 'energy' leads to conclusions that are contradicted by experiment. If we merely restrict ourselves to the two special cases examined in the preceding section we see that there is, in either case, an effective kinetic potential, proportional to \vec{u}^2 , but with a positive coefficient in one case and a negative coefficient in the other; this disqualifies it from being interpreted as an energy in the general case and blocks Rayleigh's argument.

During the 100 years that followed the publication of Rayleigh's paper the calculation has been repeated in numerous textbooks, including these: Chandrasekhar (1955 and

1980), Landau and Lifshitz (1959), Drazin and Ried (1981), Tilley and Tilley (1986), Koschmieder (1993) and Wikipedia (2017).

That Rayleigh’s prediction was contradicted by experiments must have been known to himself in 1916; that it was known to the other authors mentioned is not in doubt. Yet there is no suggestion in the later literature that Rayleigh’s argument is faulty!

Landau and Lifshitz (1959, page) claim that viscosity makes Rayleigh’s argument invalid if the cylinders rotate in opposite directions! Chandrasekhar (1981, page 275) argues that the criterion is inconsequential because the region in which it is violated is small, Drazin and Ried (1981, page 79) repeat Chandrasekhar’s argument, Koschmieder offers the most comprehensive discussion of traditional methods (*op cit* Chapter 11) but he too finds no fault with Rayleigh’s argument (*op cit* Chapter 10). Finally, Wikipedia repeats Rayleigh’s argument but offers no clue to why the criterion fails when the cylinders are rotating in opposite directions.

The Navier-Stokes equation by itself does not imply, and in fact it does not allow, the existence of an expression with the attributes of energy. A kinetic potential can be constructed in the special case of stationary, laminar Couette flow, when the velocity field is of the form (2.5); it is (Fronsdal 2014)

$$-K = \frac{b^2}{2}r^2 + ab \ln r^2 - \frac{a^2}{2r^2}, \quad (\vec{u} \cdot \vec{\nabla})\vec{u} = \vec{\nabla}K. \quad (2.8)$$

By the Navier-Stokes equation, this makes a contribution $-\vec{\nabla}K$ to the acceleration. Therefore, this function is the only kinetic potential that is consistent with the Navier-Stokes equation in the simplest case when the velocity is of the form (2.5). It is not (the kinetic part of) the Hamiltonian. *

* In *Astrophysics*, by Kippenhahn, Weigert and Weiss (212), we find definitions of normalized gravitational, internal and total energies. Then they take it for granted that the name given to the expression for the ‘total energy’ endows it with physical properties. In this manner is reproduced a correct result previously proved by Chandrasekhar (1935). Other examples of such shortcuts are legion, See e.g. Eddington (1926, page 142). Here is just one more. Arnold and Khesin in their book (1998, pages 2, 19, 37, 75, 119) lay down axioms that, for them, define Hydrodynamics. They always use the standard definition of ‘energy’, $E = v^2/2$.

III. The Action principle comes to the rescue

The Hamiltonian density is

$$h = \rho(\dot{\vec{X}}^2/2 + (\vec{\nabla}\Phi)^2/2 + \phi) + f + sT. \quad (3.1)$$

Instead of the expected square of the ‘total velocity’ we have the sum of two squared velocities. But the equation of motion, obtained by taking the gradient of (1.9), is

$$\vec{\nabla}\dot{\Phi} - \vec{\nabla} \left(-\dot{\vec{X}}^2/2 - \kappa\dot{\vec{X}} \cdot \vec{\nabla}\Phi + (\vec{\nabla}\Phi)^2/2 + \phi \right) = \frac{1}{\rho}\vec{\nabla}p, \quad (3.2)$$

where p is the thermodynamic pressure. Equations (3.1-2) both have the correct signs, the first gives the Hamiltonian density with the correct, positive sign for the two terms, the second equation agrees with the Navier-Stokes equation (when applicable) with the two different signs. The expression in the large parenthesis is a kinetic potential, but it is not an energy density.

The Navier - Stokes equation combines the two types of flow in a single velocity field and deals correctly with both of them, but that is as far as one can go with a single velocity field, for the irrotational flow is Eulerian and the solid-body flow is Lagrangian.

We conclude that the action principle is not in conflict with the traditional treatment of this type of flow, but it completes it by giving us the equation of continuity, as well as an energy density; the Hamiltonian is a first integral of the motion and the Lagrangian, not the Hamiltonian, contains the kinetic potential.

With a Hamiltonian and a kinetic potential in hand we can apply standard methods. The first result is, of course, that a stable configuration must satisfy the Euler-Lagrange equations, making the energy stationary with respect to all perturbations. A deeper analysis studies harmonic perturbations to first and second order of perturbation theory, as in Chandrasekhar (1955 and 1980), Landau and Lifshitz (1959), Drazin and Ried (1981) and Koshmieder (1993).

In the presence of viscosity there can be no energy conservation and no action principle. But there is a standard and natural way to modify the Euler-Lagrange equations. Instead of (2.3) one poses

$$\frac{d}{dt}\vec{m} = \bar{\mu}\rho\Delta\vec{v}. \quad (3.3)$$

We have limited our attention to stationary flows with $\Delta\vec{v} = 0$, taking our inspiration from the traditional point of view.

IV. Stability of laminar Couette flow by the action principle

According to (3.2), the kinetic potential is the function

$$K = -\dot{\vec{X}}^2/2 - \kappa\rho\dot{\vec{X}} \cdot \vec{\nabla}\Phi + \vec{\nabla}\Phi^2/2. \quad (4.1)$$

We shall now assume, as did Rayleigh, that a necessary condition for stability is that $K' = \partial K/\partial r$ is negative, which is equivalent to the condition that the gradient of the pressure points outwards.

By the gauge constraint, for horizontal, circular flow there are constants a, b such that

$$-\vec{\nabla}\Phi = \frac{a}{r^2}(-y, x, 0), \quad \vec{m} = -\vec{\nabla}\tau = \frac{b\kappa}{r^2}(-y, x, 0). \quad (4.2)$$

The second formula solves the gauge constraint. The velocity of mass transport is

$$\vec{v}\kappa\frac{\vec{m}}{\rho} - (1 + \kappa^2)\vec{\nabla}\Phi = \left(\frac{\kappa^2 b}{r^2\rho} + (1 + \kappa^2)\frac{a}{r^2}\right)(-y, x, 0) = \frac{1 + \kappa^2}{r^2}\left(\frac{c^2 b}{\rho} + a\right)(-y, x, 0), \quad (4.3)$$

with $c^2 := \kappa^2/(\kappa^2 + 1)$. The only constraint on the field \vec{v} is that it must be harmonic, thus

$$\frac{1}{\rho} = 1 + \alpha(1 - r^2), \quad \alpha \text{ constant}. \quad (4.4)$$

We have said that the pressure must increase outwards. For any normal fluid, for which the adiabatic derivative $dp/d\rho$ is positive, this implies that the density must increase outwards, so the constant α must be positive. (We have normalized the density at the outer boundary to unity.) This relates α to ρ_i ,

$$\alpha = 4.54(1/\rho_i - 1). \quad (4.5)$$

In terms of these vector fields

$$-K = \frac{\vec{m}^2}{2\rho^2} - (\kappa^2 + 1)\vec{\nabla}\Phi^2/2 = \frac{1 + \kappa^2}{2r^2} \left(\frac{b^2 c^2}{\rho^2} - a^2 \right), \quad c^2 := \frac{\kappa^2}{1 + \kappa^2}. \quad (4.6)$$

By (4.5),

$$-K = \frac{1 + \kappa^2}{2} \left(b^2 c^2 \left(\frac{(1 + \alpha)^2}{r^2} - 2\alpha(\alpha + 1) + \alpha^2 r^2 \right) - \frac{a^2}{2r^2} \right) \quad (4.7)$$

and

$$-K' = \frac{1 + \kappa^2}{r} \left(\frac{a^2}{r^2} + b^2 c^2 \left(\alpha^2 r^2 - \frac{(1 + \alpha)^2}{r^2} \right) \right). \quad (4.8)$$

Now that we have a kinetic potential we can apply Rayleigh's criterion, $K' < 0$.

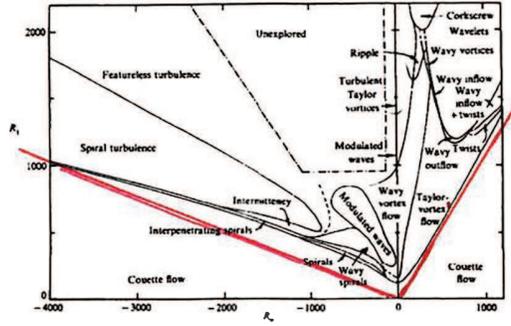


Fig.4.1. Two straight through the origin are the best possible fit of a locus $K' = 0$ to the "hyperbolic" curve that has been observed.

Boundary conditions

The walls of the two cylinders move with angular velocities

$$\omega_i \hat{\theta} = \frac{\omega_i}{r}(-y, x, 0), \quad \omega_o \hat{\theta} = \frac{\omega_o}{r}(-y, x, 0) \quad \omega_i, \omega_o \text{ constant.} \quad (4.9)$$

The velocity of mass transport is \vec{v} , Eq.(4.3), this is the velocity that must satisfy the no-slip boundary conditions, whence

$$\frac{1 + \kappa^2}{r_i^2} \left(a + \frac{bc^2}{\rho_i} \right) = \omega_i, \quad \frac{1 + \kappa^2}{r_o^2} \left(a + \frac{bc^2}{\rho_o} \right) = \omega_o, \quad (4.10)$$

and

$$a = \frac{r_o^2 \rho_o \omega_o - r_i^2 \rho_i \omega_i}{(\rho_o - \rho_i)(\kappa^2 + 1)}, \quad b = \rho_o \rho_i \frac{r_i^2 \omega_i - r_o^2 \omega_o}{\kappa^2 (\rho_o - \rho_i)}. \quad (4.11)$$

When these quantities are expressed in terms of the angular velocities we see that each of the loci $K = 0$ and $K' = 0$ consist of 2 straight lines through the origin of the angular velocity diagram. The best result that can be obtained with the assumption that K' must be negative, is for the lines $K' = 0$ coincide with the 'asymptotes' of the experimental curve, as in Fig.4.1. That would leave a substantial region that is stable, although K' is positive.

But even that may not be possible.

The zero locus of (4.8) consists of the two lines

$$\frac{a}{bc} = \pm \beta, \quad \beta = \sqrt{(1 + \alpha)^2 - \alpha^2 r^4},$$

or

$$\frac{\omega_i}{\omega_o} = \frac{1 \pm c/\beta r^2 \rho_i}{1 \pm c/\beta r^2}. \quad (4.12)$$

If the two asymptotes are in the first and fourth quadrant it follows that either

$$c < \beta < c/\rho_i, \quad \text{or else} \quad c > \beta > c/\rho_i.$$

1. In the first case $\rho_i < 1$. Then $\alpha > 0$. At the outer boundary

$$\beta^2 = 1 + 2\alpha = 1 + 2\frac{1/\rho_i - 1}{1 - r_i^2} < 1/\rho_i^2.$$

By (4.12), $\beta^2 = 1 + 2\alpha < c^2/\rho_i^2 < 1/\rho_i^2$, which reduces to $2/(1 - r_i^2) < 1 + 1/\rho_i$, which is false since the density is very close to 1. At the inner surface, β is even larger.

2. In the second case $\rho_i > \rho_o$, which is anti-intuitive, but perhaps our intuition needs to be re-educated.

V. A proposal

It is known that some types of instability are accompanied by bubble formation. We suggest that this may be related to local evacuation (or more generally to a physical breakdown of the laminar nature of the flow) and that it happens at a particular value of the chemical potential, the density and of the corresponding value the kinetic potential K .

We shall test this hypothesis. We are not rejecting the idea that K' must be negative; it turns out that it is negative almost everywhere.

As we see from Eq.(4.7), a locus of $K = 0$ consists of two straight lines. At the inner boundary,

$$\frac{\omega_i}{\omega_o} = \left(\frac{r_o}{r_i}\right)^2 \frac{1 \pm c}{1 \pm c(\rho_i/\rho_o)} \quad (5.3)$$

and these must have opposite signs, hence $1 > c > \frac{\rho_o}{\rho_i}$. At the outer boundary,

$$\frac{\omega_i}{\omega_o} = \left(\frac{r_o}{r_i}\right)^2 \frac{1 \pm c(\rho_o/\rho_i)}{1 \pm c}$$

and $1 > c > \rho_i/\rho_o$. This implies that, as the velocity is increased, the instability first manifests itself at the boundary with the higher density.

The value of K varies with the radius r . If, as the velocity is increased, the instability makes a first appearance at the boundary B , the inner or the outer cylinder, then we expect that the parameters can be tweaked to make a locus $K(r_i) = A$ or $K(r_o) = A$ an adequate approximation to the experimental limit curve and that the family of loci, $K(r) = A, r_i < r < r_o$ displace upwards as we move away from B .

We proceeded as follows.

(1) We first examined the possibility that the density is highest at the outer boundary, which implies that the density is higher there, as had been expected.

(2) Then we repeated the exercise at the inner boundary, in which case the density is higher there, against expectations.

The first step was a total failure; we shall not say more about it. The second step, conducted in the same way, will now be described in detail.

Main results

Assume that $\rho_i > \rho_o$. Fig. 5.1 shows a random set of loci of K . Fig. 5.2, taken from Andereck *et al* (1986) is a summary of their results, the lowest line, resembling a hyperbola, is the upper limit of observed, basic Couette flow.

Fig. 5.1. The curves are K loci.

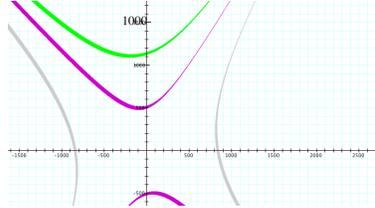


Fig 5.3 shows a K locus that best fits the experimental curve, with $\rho_o/\rho_i = .9$, $\kappa = 5.3$ and $R = r_i$. Identical curves are obtained with $\rho_o/\rho_i = .99$, $\kappa = 17.5$, $\rho_o/\rho_i = .999$, $\kappa = 55$ and so on, calculated up to $\rho_o/\rho_i = 1 - 10^{-6}$. The slope of the right asymptote is predicted without ambiguity to be close to $(r_o/r_i)^2$, – a bullseye for the theory. See Eq. (5.3). The slope on the left is very sensitive to the value of κ ; this too can be seen by inspection of (5.3).

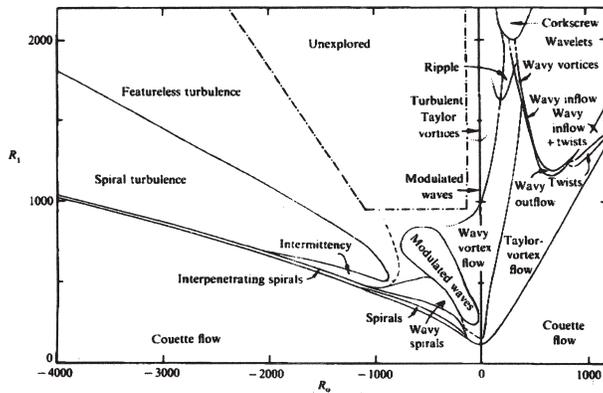


Fig. 5.2. Experimental results of Andereck et al. The abscissa (ordinate) is the angular velocity of the outer (inner) boundary. The lower ‘hyperbola’ is the upper limit of stability of laminar Couette flow.

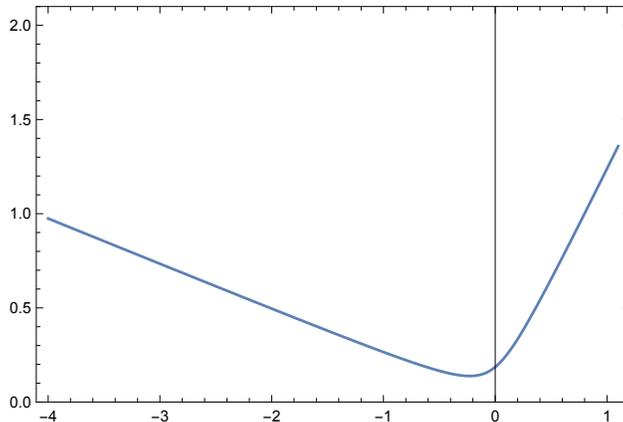


Fig.5.3. A locus $K(r_i) = A > 0 = \text{constant}$. The lines $K = 0$ lie below.

The subsequent figures show the result of repeating the calculation, with the same values of the parameters and the same value of A , at several points in the interval $r_i < r < r_o$.

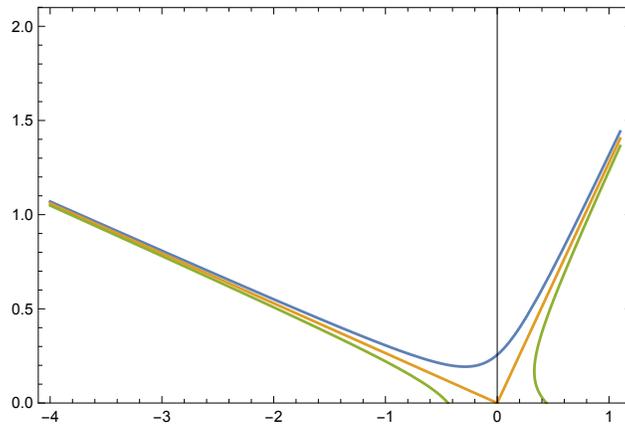


Fig.5.4. Loci $K = A_- < 0, 0, A_+ > 0$ at the inner boundary, $r = r_i = .883$. The three curves are drawn for 3 values of $K = A$, the same values in all the diagrams that follow.

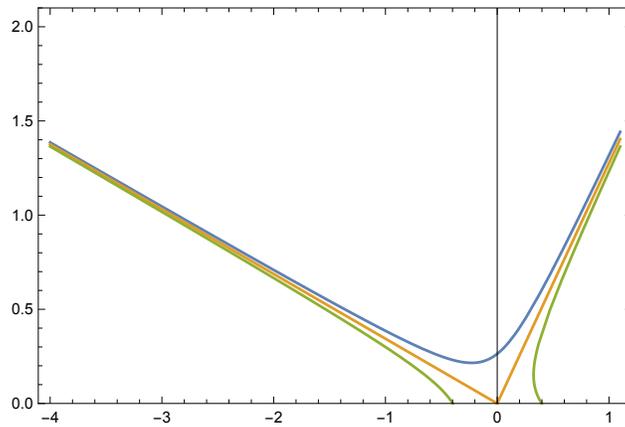


Fig.5.5. The locus $K = A$ with the same values of A at $r = .888$.

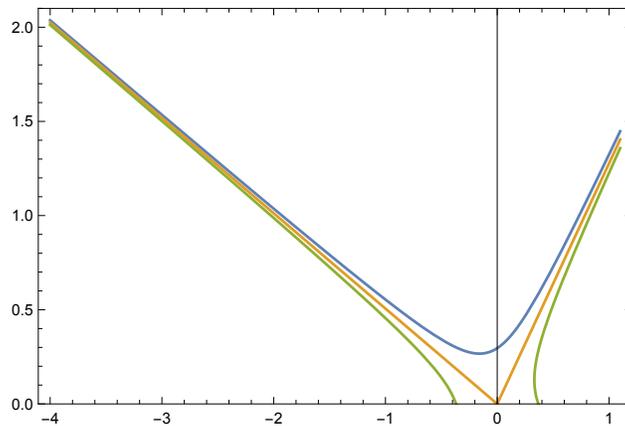


Fig.5.6. At $r = .893$. the locus is moving upwards, almost everywhere.

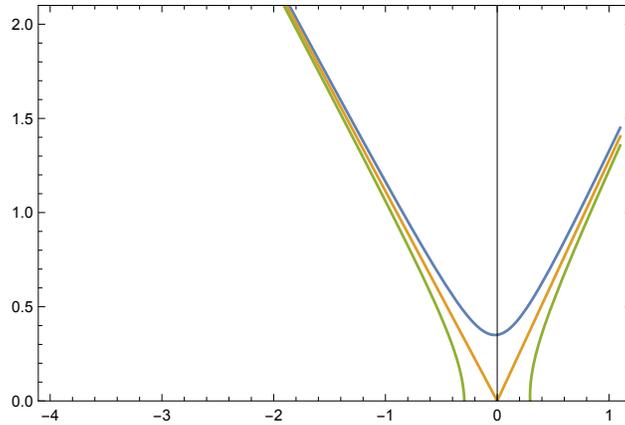


Fig.5.7. At $r = .898$.

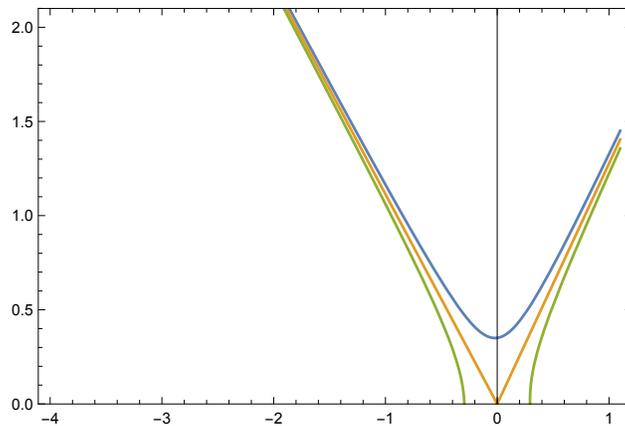


Fig.5.8. At $r = .92$.

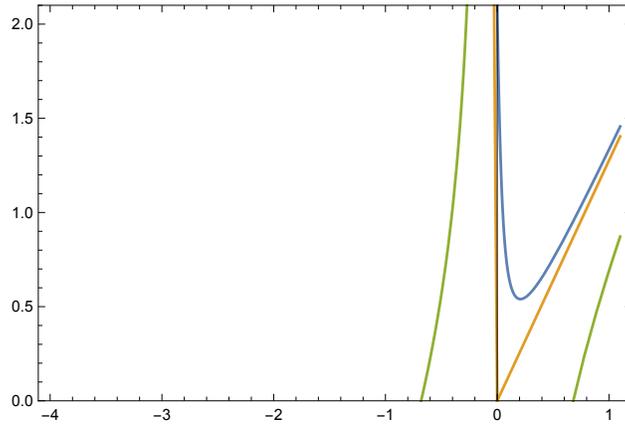


Fig.5.9. And at $r = .98$.

In each of these diagrams two straight lines through the origin make up the $K = 0$ locus. The hyperbola that lies between them is a locus of $K = A_+ > 0$. The two lines that lie outside form a locus of $K = A_- < 0$. All these figures have the same parameters and the same values of A_{\pm} . The series start at the inner boundary and proceeds to the outer boundary.

The crucial question is what happens to the locus $K(r) = A$ as r is increased, for fixed

values of the parameters and for a fixed value of A . Comparing Figures 5.5-10 it can be seen that, almost everywhere, the locus moves upwards with increasing radius; hence at any point in the diagram, the value of K decreases. Only the region near the right asymptotes is difficult to assess.

To get a different view of the situation we next plot the zero loci of K' , with the same parameters. As the result is virtually independent of the radius in the range $r_i < r < r_o = r_o$; we show only the case $r = r_i$. It is seen that K' is negative everywhere except in a small region on the right, which confirms what we concluded by visual inspection of the K loci.

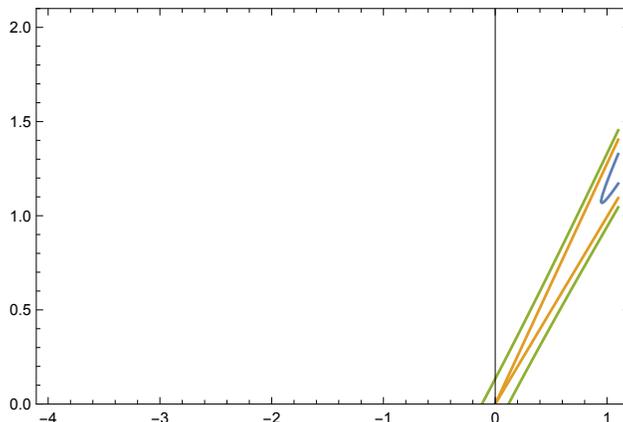


Fig. 5.11. These are loci of the derivative of the the kinetic potential. It is zero on the two straight lines through the origin and it is negative between them, everywhere else it is positive.

That is reassuring, except for the fact that the pressure gradient has the ‘wrong’ sign, for balance the centrifugal force should have the wrong sign as well. Instead of balancing each other, the centrifugal force and the pressure gradient reinforce each other.

If the (apparent) behavior of the system is contrary to our intuition, then one thing that can be done is to educate our intuition.

Are we forgetting a third force? Yes, and perhaps a fourth.

Surface tension

The success of the no-slip boundary condition is witness to a considerable surface tension, due to adhesion of the fluid to the walls. Directly, this affects only the first layer of molecules near the walls, but water has a considerable tensile strength, as is known from the ease at which it withstands negative pressure.

At the beginning of the experiment the fluid is at rest and the density is uniform. Then, as we begin to rotate the cylinders the adhesion of the fluid to the wall exerts a force. By the principle of Le Chatelier, the reaction of the system is always to resist; hence, at least at first, the pressure and the centrifugal effect must cooperate against the effect of the adhesion. The non-slip boundary condition is not just a mathematical rule; it speaks for the action of a third force. In our model, the kinematic potential and the pressure continue to act together in most of the stable configurations.

Consider two volume elements, one on the inner cylinder and the other at the outer cylinder, and a thin smooth tube connecting the two points. Make this more precise by assuming that the tensile strength of the fluid is transmitted along the tube. Then there is one case when this force is constant, when the instantaneous velocity of the two endpoints are equal. In that case $\omega_i/\omega_o = \rho_o/\rho_i$, which defines the limit of stability in the first quadrant. At that point the centrifugal force has to cancel the force produced by the gradient of the pressure; the pressure increases outwards, which is exactly what Fig.5.4 tells us.

Perhaps that is what we see when we notice that K' turns positive along a radial line on the right of the diagram.

These diagrams show that the domain of stability gradually expands as we move outwards, K' is negative almost everywhere and this is a necessary condition for stability.

Entropic forces

They have been completely ignored until now, in obedience to a popular and useful strategy: Assume, as a working assumption, that the specific entropy density is uniform; that allows the hydrodynamic model to be lifted smoothly to thermodynamics *, which lets us calculate the temperature and the thermodynamic potentials once we have chosen an expression for the free energy density. If this works, fine, if not there may be entropic forces at play.

In the present case we have followed this protocol and it has not led to any difficulties, the experimental data having been succinctly explained to considerable accuracy. This not so much due to the superior quality of the model as to the paucity of experimental data. This may change. The system is not at equilibrium; some energy is being supplied, there is likely to be a temperature gradient and there may be some heat flow. The success of the working assumption is not fully explained.

VI. Conclusions

Energy, with associated conservation laws, is the very soul of theoretical physics. It is entirely natural that, in the many contexts where it is not available, attempts are made to invoke it anyway. But our efforts are better spent if we strive to formulate all of physics in terms of action principles. The difficulty, of course, is to determine the action or, before that, to find the variables that make it possible.

The quest for action principles for hydrodynamics and thermodynamics was pursued most vigorously during the late 19'th and early 20'th century, by some of the most important physicists of that pioneering age: Helmholtz, Maxwell, Cartan, Einstein and others. In 1950 John von Neuman wrote:

In a memorandum to O. Veblen, dated March 26, 1945, John von Neuman (1945) laments the fact that “hydrodynamical problems, which ought to be considered relatively simple, offer altogether disproportionate difficulties”; he says “ that the true technical reason appears to be that variational methods have ... hardly been introduced in hydrodynamics.” And he adds: “It is well known that they could be introduced, but what I

* Fronsdal 2011

would like to stress is that they have not been used to any practically important scale for calculations in that field”.

The reason that they had not been used is to some extent explained by the internal inconsistencies related to the use (or misuse) of the energy concept.

A viable action principle for rotational hydrodynamics, thermodynamics and General Relativity is now available (Fronsdal 2015, 2017). It is not unique, but it is the most economical; complicated systems require more variables. This paper offers a first instance of an application, to a system that has resisted a complete development even in the special case of incompressible flows. An application to planetary systems, in the Newtonian approximation, is in the works and the lifting of this model to General Relativity needs only dedication.

Future experiments

In the present contexts of elementary hydrodynamic systems much work has been done on incompressible flows. The more challenging problem of compressible flows require measurements of temperatures and pressures, which are difficult. We need more measurements of this type and we need experiments with wider horizons. Andereck and others have explored the full range of flow regimes in the Couette problem, we need to vary the fluid (even to the extent of using gases) as well as the density of absorbed air, ambient temperature and pressure and, especially, surface tension.

Appendix: A theory of incompressible fluids

We set

$$\frac{\rho_i}{\rho_o} = 1 + \xi, \quad c = 1 - \psi, \quad 0 < \psi < \xi$$

and obtain, on the asymptotes $K = 0$, in the first quadrant,

$$\frac{\omega_i}{\omega_o} = \left(\frac{r_o}{r_i}\right)^2 \frac{2 - \psi}{2 - \psi + \xi} \approx 1.28$$

and in the fourth quadrant,

$$\omega_i/\omega_o = \left(\frac{r_o}{r_i}\right)^2 \frac{\psi}{\psi - \xi} = -1.28 \frac{1}{\frac{\xi}{\psi} - 1} < -1.28.$$

In the best fit $\xi/\psi \approx 5$.

We get a model of an incompressible gas by taking the limit of large κ . In this case it is better to renormalize the field \vec{X} by absorbing a factor κ , to express the the Lagrangian density (1.5) as

$$\mathcal{L} = \rho(\dot{\Phi} + \dot{\vec{X}}^2/2\kappa^2 + \rho\dot{\vec{X}} \cdot \vec{\nabla}\Phi - \vec{\Phi}^2/2 - \phi) - W[\rho]. \quad (6.1)$$

The second term may perhaps be neglected in some applications.

The important fields are the velocity of mass flow

$$\vec{v} = \dot{\vec{X}} - \vec{\nabla}\Phi,$$

and the renormalized vector field

$$\frac{\vec{w}}{\kappa} = \frac{1}{\kappa^2} \dot{\vec{X}} + \vec{\nabla} \Phi.$$

In the limit $1/\kappa \rightarrow 0$ the field \vec{X} becomes a non-dynamic field, as in the approach initiated by Hall and Vinen *op cit*.

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