

## VIII. Superfluid Helium

### VIII.1. Experiments and interpretation

Here is a brief summary of the first important experiments on **superfluid Helium**, the development that culminated with Landau's two fluid theory. It is natural that we should take a special interest in this system, for it stands out as the most important example of two-fluid thermodynamics. This chapter is written under the perspective of potential hydrodynamics, consequently there is no reference to two kinds of flow, to phonons and rotons. The development of Conservative hydrodynamics, in Chapter X, will allow us to deal with those aspects.

### The fountain experiment

The story of superfluid Helium started with revolutionary experiments by Kapitza (1938) and by Allen and Misener (1939). Liquid Helium is cooled in vessels  $A$  and  $B$ ; initially the level is higher in  $A$ . A capillary that connects vessel  $A$  to the vessel  $B$  is very fine and no significant amount of liquid gets through, until the temperature descends below 2.19K. Both vessels are then thermally isolated. Subsequently, if a temperature difference is established, with  $T_A > T_B$ , then the level in vessel  $B$  will rise due to a flow through the tube, Fig. 8.1.1, until the pressure due to gravity provides a sufficient counter-force and an equilibrium is established. The mass flows toward the lower temperature and the temperature difference increases.

The effect is quite dramatic: The flow is independent of the pressure and, what is more, it is nearly independent of the diameter of the tube though this be varied as much as by a factor of 50!

This highly unusual behavior led F. London (1938) to suggest that the fluid passing through carries no entropy. At about the same time L. Tisza (1938) suggested that liquid Helium below 2.19 degrees Kelvin is a mixture of two components or phases, a normal component and a **superfluid component** with two highly unusual properties. It has no viscosity, which helps it get through the tube, and it carries no entropy with it, so that no heat transfer takes place. This is known as the 2-fluid model.

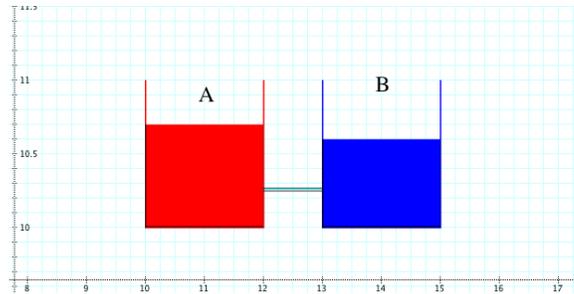


Fig.8.1.1. The fountain effect. The central portion (light blue) is filled with a fine powder that is impermeable to Helium liquid helium at temperatures above the  $\lambda$  point.

In 1945 Landau, in a very imaginative analysis of this experiment suggested an alternative interpretation; instead of two fluids he introduced 2

kinds of flow. Let us call this the two-flow theory. For our purpose it is natural to look at the problem as a competition between two theories: two fluid or two flows. Historically what emerged was a compromise that adopted elements from both. The idea of two fluids is still thought to be essential, although Landau's warning against postulating that there are two kinds of Helium atoms suggests that it not be taken literally (See *e.g.* Tilley and Tilley 19??).

Further variations of this type of experiment are described and interpreted in lucid detail in a review by Roberts and Donnelly (19??). The arrangement of 2 vessels connected by a tube so thin that only the superfluid gets through is the same, but instead of heating one of the vessels it is raised to a higher level. The difference in gravitational potential gives rise to a flow from the higher to the lower vessel, with the same unusual characteristics as in the original experiments. What is more, this process is completely reversible. On the basis of this experiment, and others to be described later, it is established that, for a fixed temperature and pressure, only one equilibrium concentration is possible, only one equilibrium configuration exists for given total density and temperature.

And yet the result is not a separation of the two components: with the establishment of a new **equilibrium, the concentration** in either vessel is determined uniquely by the temperature and the pressure. It is impossible to produce a mixture, at the same temperature and pressure, with a different concentration. The equilibrium concentration is not an independent variable.

This conclusion is highly significant and difficult to reconcile with the interpretation of other experiments. We shall not describe these other processes here, but we are referring to statements of type "the superfluid seems to flow through the normal fluid without resistance". If only one concentration is possible for any fixed value of temperature and pressure, then it is difficult to understand what it means that the superfluid flows through the normal fluid. Actually, this difficulty arises when we regard an equilibrium configuration as a snapshot of a quasi static sequence that involves flow, as would seem to be justified in the case of flows at very low velocity. We seem to be in a situation that resembles the equilibrium in the dissociating mixture of atomic and molecular gases that we have examined in Section IV.3.

At this stage we do not yet have an understanding of hydrodynamics that

allows us to examine Landau's two-flow theory; for this reason we shall not discuss experiments that involve flow. Instead we shall pursue the idea that Helium below the  $\lambda$  point is a mixture of two fluids, a normal component and a superfluid component, with an interaction that transforms a quantity of the one into the same quantity of the other, preserving mass

$$\begin{array}{ccc} \text{Normal} & \begin{array}{c} \leftarrow \\ \rightarrow \end{array} & \text{Super} + \epsilon, \end{array}$$

where  $\epsilon$  is the difference in energy per unit of volume, assumed constant.

That the **entropy of the super fluid** component is near zero (supposing that it is) is a reflection of the excitation spectrum of the Helium atom; this information can be used but not justified within our context. It suggests that the free energy of the super fluid component is independent of the temperature, which makes sense when the total free energy is regarded as a sum  $f_1 + f_2$  of those of the two separate gases. A reasonable inference is that the temperature of the liquid in equilibrium cannot be lowered without converting some of the liquid from normal fluid to superfluid. And this is what is observed.

The underlying, atomic description of what may be going on will not be discussed in this book. The (by now) standard explanation is that the individual Helium atom has a single, isolated ground state of very low energy. When the temperature is lowered the atoms, being bosons, all seek the lowest level, and since there is only one isolated level at this lowest energy, the effective number of degrees of freedom is zero and there are, effectively, no populations of the higher energy levels. (Bose-Einstein Condensation.) According to what is now generally accepted, fluid divides into two parts, with the atoms of one, the 'superfluid', condensing into the lowest level. London speaks of a phase separation in momentum space and Landau later advanced the same concept.

But there is a contrary opinion. Landau, in his first paper on this subject, says that it would be wrong to think of two different types of atoms; instead, he writes, there are two different types of flow, referred to as **phonons and rotons**. Phonons are whorls of **locally irrotational flow**, rotons are said to carry vorticity; they are also assigned the role of flows of the solid body type. This statement by Landau is repeated in the last edition of the *Fluid Mechanics* by Landau and Lifshitz (1969). The unusual behavior does not by itself imply that liquid Helium is governed by a different kind of thermodynamics; we shall see, in Section 8.2.3, that London's intuition about the

entropy is sufficient to explain some of the unusual behavior of this liquid, so long as we limit our attention to quasi-static phenomena.

The theory of flows to be developed later will make this question of interpretation much more acute and much more complicated. We shall not say more about that problem at this place.

### The entropy

We have proposed the representation

$$s = \rho_1 S_1 + \rho_2 S_2, \quad (8.1.1)$$

of the **entropy functional**; (density) of a mixture. The specific entropies  $S_1$  and  $S_2$  are adiabatic invariants. In the analogous case of saturation, heating results in an increase of the amount of gas while the specific entropies  $S_1$  and  $S_2$  remain fixed or almost so.

This formula appears to help us understand the fountain experiment. As was discovered in Chapter V, if each component is treated as if it has no interaction with the other, the equations of motion for a system at rest, namely

$$\dot{v}_i = \frac{\partial(f + sT)}{\partial \rho_i}, \quad i = 1, 2,$$

can be expressed in the form (when  $S_2 = 0$ )

$$\dot{v}_1 = -\frac{1}{\rho_1} \vec{\nabla} p_1, \quad \dot{v}_2 = -\frac{1}{\rho_2} \vec{\nabla} p_2 - S_1 \vec{\nabla} T. \quad (8.1.2)$$

The last term represents a force that drives the superfluid component into the connecting tube, just as in Landau's theory. The difficulty with this argument is, as we have said, that the 'partial pressures are not defined.

Very precise measurements of the entropy density  $s$  are available. In the upper half of the temperature interval, where the normal gas dominates, the first term is expected to make the larger contribution and on this assumption it is seen that the specific entropy  $S_1$  varies only slowly with the temperature, which confirms the view that the lowering of entropy takes place, mainly, through conversion of the normal fluid to superfluid. But the measurement of partial densities is indirect and depends on the interpretation.

**TABLE 8.1**

$$\begin{aligned}
 p = 00.0, & \quad s = 1.59\rho_1 + .0058 T^3 \rho_2, \\
 p = 05.0, & \quad s = 1.45\rho_1 + .0042 T^3 \rho_2, \\
 p = 10.0, & \quad s = 1.36\rho_1 + .0025 T^3 \rho_2, \\
 p = 15.0, & \quad s = 1.289\rho_1 + .0025 T^3 \rho_2, \\
 p = 20.0, & \quad s = 1.18\rho_1 + .0020 T^3 \rho_2, \\
 p = 25.0, & \quad s = 1.129\rho_1 + .0017 T^3 \rho_2.
 \end{aligned}$$

Table 8.1. Relation between measured entropy density and the equilibrium values of densities and temperature, covering the whole interval from .1K to 2.1K. Note that entropies given by Brooks and Donnelly (1977) are specific densities. Our “s” is density in  $J/cm^3$ ; the entropy quoted by Brooks and Donnelly multiplied by mass density. In the table the pressure is given in atmospheres.

There is no significant dependence of the specific entropy  $S_1$  on the temperature when  $1 < T < 2$ . Towards still lower temperatures, when the normal gas is absent, the entropy of the super fluid becomes dominant and there is no way to go except to reduce the specific entropy  $S_2$ . We see from the Table that it is proportional to the third power of the temperature.

Note that, experimentally, the entropy of the superfluid is not zero, though small, but that its dependence on the temperature is very different from  $S_1$ .

### The equilibrium configuration

A relation between the basic variables  $\rho_1, \rho_2$  and  $T$  defines the equilibrium configurations of the system. Using data found in Putterman’s book we plot all available physical points in the interval  $1.3 < T < 2.1$  in Fig.8.1.2 and find that the following equation describes it well,<sup>1</sup>

$$Q(\rho_1, \rho_2, T) := (\rho_1 + a\rho_2 + b)(\rho_1 + c\rho_2 + d) + 10^{-4} = 0,$$

with

$$a = -.15, \quad b = .014 - .21(T - 1.3)^2, \quad c = 1.86 - .31T, \quad d = -.398 + .113T.$$

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<sup>1</sup>See however recent experimental work on superfluid Helium at negative pressure.

Using data collected by Brooks and McConnelly we get a slightly different result, see Fig.8.1.3. A more convenient expression for the equilibrium relation between  $\rho_1, \rho_2$  and  $T$  will be found later.

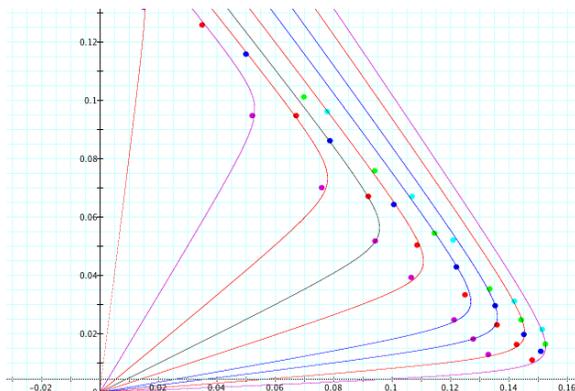


Fig.8.1.2. The set of equilibrium configurations as given by Putterman. The ordinates are the two densities. The color goes with the pressure;  $p = 0$  for the left most purple dots and  $p = 25atm$  for the (also purple) dots at the right end. The lines connect points with the same temperature, 1.3 K for the outer line and 2K for the inner line. The extrapolation from the purple dots to the origin (negative pressure) is not justified by the experimental data.

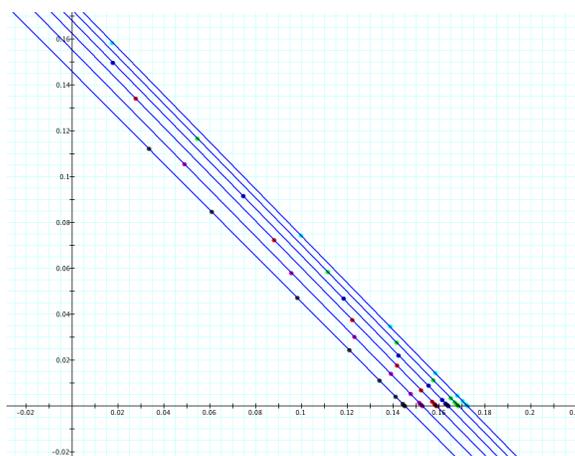


Fig.8.1.3. The set of equilibrium configurations according to Brooks and Donnelly. Blue lines are loci of equal pressure.

The data points used are tabulated at the end of this chapter.

Values of the two densities can be represented with high precision as follows, for each of six values of the pressure, the densities in  $mg/cm^3$  and the pressure in atmospheres. The variation with pressure, though relatively weak, makes this representation inconvenient.

**TABLE 8.1.2**

$$\begin{aligned}
 p = 00, & \quad \rho_1 = 1.90 T^{5.5}, & \rho_2 = 145.0 - 1.74 T^{5.6}, \\
 p = 05, & \quad \rho_1 = 2.17 T^{5.6}, & \rho_2 = 152.5 - 1.86 T^{5.8}, \\
 p = 10, & \quad \rho_1 = 2.56 T^{5.7}, & \rho_2 = 159.0 - 2.20 T^{5.9}, \\
 p = 15, & \quad \rho_1 = 3.03 T^{5.8}, & \rho_2 = 164.0 - 2.65 T^{6.0}, \\
 p = 20, & \quad \rho_1 = 3.70 T^{5.9}, & \rho_2 = 168.5 - 3.20 T^{6.1}, \\
 p = 25, & \quad \rho_1 = 4.40 T^{6.0}, & \rho_2 = 172.5 - 4.00 T^{6.2},
 \end{aligned}$$

Table 8.1.2. This representation of the concentration is valid over the whole temperature interval, from .1K to 2.1K. Later we shall see how it was measured.

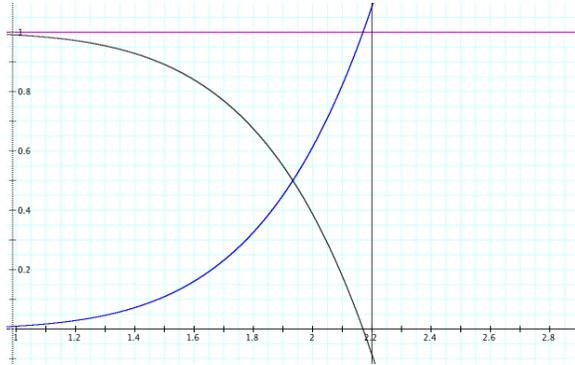


Fig.8.1.4. Sketch of the concentrations of the normal fluid (increasing) and the superfluid component (decreasing) between  $T = 1K$  and  $T = \lambda$ , for fixed pressure. The actual measurements are indirect; one method relies on an interpretation of the cross section for **neutron scattering**.

### Second sound

Landau's theory is distinguished by the fact that it treats the two components of a mixture as independent degrees of freedom: two densities and two velocities. It predicted the existence of two kinds of sound, a normal mode (first sound) in which both oscillations are in phase and another, second sound, in which they are in opposite phase. This was a real prediction, made before any experiments confirmed the existence of two sounds in liquid Helium. This feature of Landau's theory appears to contradict the conclusion that we had reached earlier: that there is only one equilibrium mixture at each temperature. A possible resolution of this difficulty may be that both flows are preserved in the bulk; transformations take place only with a transfer of heat, as in collisions with the walls. A true equilibrium is reached only after this process has had time to adjust the concentration. The value of the equilibrium concentration would be established by the process of collisions with the wall. It would be reasonable to ask for a more convincing explanation.

Having postulated the existence of two sound modes we can affirm with some assurance that there must be two densities and **two velocities**. With the present limitation to irrotational vector fields the kinetic part of the Lagrangian density is

$$\rho_1(\dot{\Phi}_1 - \vec{v}_1^2/2 - \phi) + \rho_2(\dot{\Phi}_2 - \vec{v}_2^2/2 - \phi). \quad (8.1.2)$$

The gravitational potential  $\phi$  will be neglected. The appearance of an additional degree of freedom naturally implies that two sound modes are likely to exist; the surprise is not the existence of two sounds in liquid Helium but the fact that the observation of a similar phenomenon is rare in other mixtures. The reason, as was seen, is that most mixtures are characterized by a strong interaction between the components, interactions that effectively quash the mode in which the two fluctuations are in opposite phase. Other materials do not exhibit these special features because at comparable temperatures they are in a solid state.

Unless strong interactions between the two components prevent it, two sounds come naturally.

An implication of this is that we shall have two equations of continuity,

$$\dot{\rho}_i + \text{div}(\rho_i \vec{v}_i) = 0, \quad i = 1, 2. \quad (8.1.3)$$

The observation that only the total mass is conserved suggests that there can be only one equation of continuity. But we remember that conservation

of mass is not an automatic consequence of the equation of continuity; it only follows when, in addition, one imposes the condition that the **boundary velocity** be tangential to the boundary. Here we follow Landau in that we shall impose this boundary condition on the total flow

$$\vec{j} = \sum_i \rho_i \vec{v}_i,$$

but not on the individual velocities. Both mass densities are conserved in the bulk but only the total mass is preserved in the interaction with the boundary.

In the bulk, away from the boundary, and in the absence of external manipulation of the system such as irradiation, the two liquids flow past each other, conserving their respective identities. This is the usual interpretation expressed by comments in the literature; see for example Khalatnikov (1965 page 93).

The kinetic terms in our model, including the terms  $\rho_i \dot{\Phi}_i$ , are chosen so as to make possible a Lagrangian formulation. Recall, from Section III.11, that the inclusion of the velocity potential helps to make the Lagrangian invariant under **Galilei transformations**. Khalatnikov, in a book that has become a classic reference (Khalatnikov 1965) also aimed at a Lagrangian formulation, but without the help of the velocity potentials he was constrained to reduce the kinetic energy to the invariant expression  $\rho(\vec{v}_1 - \vec{v}_2)^2/2$ . According to Putterman (1975), there is an analogous contribution to the pressure. Our Lagrangian is Galilei invariant and we do not adopt these measures.

### Thermodynamic pressure

The thermodynamic pressure is defined as

$$p = \rho_1 \frac{\partial f}{\partial \rho_1} + \rho_2 \frac{\partial f}{\partial \rho_2} - f. \quad (8.1.4)$$

The free energy is a function of the three independent variables  $\rho_1, \rho_2$  and  $T$ . It is common usage in thermodynamics of mixtures since Gibbs, and especially so in Landau's treatments of liquid Helium, to reduce the concentration to the role of a parameter, denying it the respect that it deserves as a thermodynamic variable. Such treatments are dynamically incomplete, being partially on shell.

The pressure on the wall has an additional contribution whenever the wall is maintained at a temperature that is different from the fluid. In the case of a heated wall, heat is transmitted to the liquid by the conversion of superfluid to normal fluid. This upsets the balance and, to compensate, normal fluid flows towards the wall and super fluid flows away from the wall. The additional pressure has been measured (Hall, 1954). The standard interpretation of this effect gives results that are in qualitative agreement with experiment.

The on shell pressure can be expressed in terms of  $T, \rho_1$  and  $\rho_2$ , this relation is found by measurements at equilibrium to take the following form; for example,

$$T = .1 : \quad p = 25 - 1360(.17246 - \rho) + 14300(.17246 - \rho)^2. \quad (8.1.5)$$

See Fig. 8.1.3 where the nearly exact dependence of the pressure on the total density  $\rho = \rho_n + \rho_s$  alone is manifest. The derivative,

$$\frac{dp}{d\rho} = 28600\rho - 3.6384 = c^2$$

varies from 526.7 at  $p = 0$  to 1308.8 at  $p = 25$ . The unit of pressure is one atmosphere. To convert to cgs units multiply by  $10^5$  to obtain the square of isothermal velocity of normal sound in cm/sec. At  $T = .1$ ,  $c = 229m/sec$  at  $p = 0$  and  $c = 362m/sec$  at  $p = 25$ . At higher temperatures much the same. The quasi independence of the speed on the temperature is significant.

### Critical behavior

According to Arp (1989), all singular features of the thermodynamic functions can be fitted with a specific heat that is singular at  $T = \lambda$ . A term like

$$(T - \lambda) \ln\left(1 - \frac{T}{\lambda}\right)$$

in the free energy will produce just such a term in the heat capacity. Al-

though we shall use a similar phenomenological function, we are not going to take the point of view of Arp, who emphasized this singular behavior in a trial function with 41 adjustable coefficients. Arp's very thorough work suffers from an important omission: it deals with a single fluid. There is no account of the separate densities of the two components, consequently no

mention of second sound. All thermodynamic relations ignore the concentration.

Let us remember that van der Waal’s highly successful modeling of a gas near and above the critical gas/liquid point did not predict the “correct critical behavior”. This refers to a very small region that involves altogether different concepts. We do not attempt to achieve agreement with experiment in this small neighborhood.

## VIII.2. First analytic models

We shall construct idealized Lagrangian models, with features that resemble those of He II, but simplified and, in one respect, exaggerated.

### Thermodynamic variables.

We postulate that the correct choice of thermodynamic variables are  $T$ , the densities  $\rho_1, \rho_2$  and the individual, specific entropies  $S_1, S_2$ . It is by far the most convenient choice though others are of course possible. In what follows we are concerned almost exclusively with equilibrium configurations; with no flow ( $\vec{v}_1 = \vec{v}_2 = 0$ ) and all fields uniform.

### Structure of the Lagrangian.

Landau’s equations for the two fluid model was constructed as a set of equations, without a Lagrangian variational structure, with a Galilei invariant expression for “the energy density” (Putterman 1974, Khalatnikoff 1956). As we know, it is the Lagrangian that must be Galilei invariant, not the energy.

It is expected that the Lagrangian density, in a first approximation, is a sum of three kinds of contributions, the kinetic terms (8.1.2), the free energy, and the entropy term  $-sT$  with the entropy density in the form (8.1.1):  $s = \rho_1 S_1 + \rho_2 S_2$ . The principal terms in the free energy are associated with each component separately,

$$f = f_1(T, \rho_1) + f_2(T, \rho_2) + \dots ,$$

but additional terms will be needed to take interactions into account.

An adiabatic system is defined by fixing the specific entropies  $S_1, S_2$  and the pressure (or the volume). How the total entropy is apportioned between

the two species can be understood in terms of an underlying atomic theory. In the event that a successful model is found it should be reconciled with atomic theory.

### The pressure.

The term attributed to  $f$  so far, the kinetic part in Eq.(8.1.2), being linear in the density, does not contribute to the pressure (8.1.4). In He II a good approximation is (8.1.5), with variations of  $c^2$  of up to 5 percent over the range of temperatures. Our first model, before necessary improvements, has pressure

$$p = c^2(\rho - .1455), \quad (8.2.1)$$

with  $c^2$  constant. This pressure implies another term to be included in the free energy, for example

$$c^2(\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2 + .1455) \quad \text{or} \quad c^2(\rho \ln \rho + .1455). \quad (8.2.2)$$

the first expression is appropriate for two non interacting components, the other for a single component with the density  $\rho = \rho_n + \rho_s$ . Both expressions result in the pressure (8.2.1). The absence of any dependence on the temperature in this expression ensures that the velocity of propagation of ordinary (first) sound in the model has no direct dependence on the temperature, although it may depend on the concentration. In He II at equilibrium it varies from 227 to 366 m/sec. We are attempting to construct a simple model with properties that are similar to those of Helium, but not one to be identified with it.

### Entropy.

The distribution of the **entropy**— between the two species, as a function of pressure and temperature, must be determined from experiments. For liquid Helium, what is known is summarized in Table 8.1.1. Any model must make some assumption, to be justified either by experiment or by atomic theory, preferably by both. Our experience with phase transitions, dissociation and immiscible fluids is not very helpful in the present instance. The only reliable guide is observation.

We take as a principal feature of our idealized model the axiom that the measured entropy is carried by the normal component; that is,

$$s = \rho_1 S_1. \quad (8.2.3)$$

This defines a family of adiabatic systems, parameterized by a single parameter  $S_1$ . The value of  $S_1$  determines the adiabatic system and its equilibrium configuration, just as is the case for unary systems. This feature emulates one of the most dramatic aspects of He II, in an extreme form, by setting  $S_2 = 0$ , while experimentally  $S_2$  is non-zero but insignificant, except at the very lowest temperatures.

For this reason it is natural to take the temperature dependent terms in the model free energy to be independent of  $\rho_2$ . We make this another postulated property of our model. The simplest possibility is that the  $T$  dependence of  $f$  is confined to a term

$$a(T)\rho_1. \tag{8.2.4}$$

Linearity in  $\rho_1$  implies that this term makes no contribution to the pressure. The adiabatic condition is

$$S_1 = -\partial a/\partial T. \tag{8.2.5}$$

It is a feature of most familiar systems that the internal energy is first order in  $T$ . To achieve this,  $a(T)$  must include terms of the type  $T \ln T$ . Since a singularity is expected at  $T = \lambda$  we may be tempted to follow Arp, including a term  $b(\lambda - T) \ln(\lambda - T)$ , with  $b$  and  $\lambda$  constant. We leave the function  $a(T)$  undetermined for the present, but notice that the adiabatic condition makes  $T$  uniform for all adiabatic processes.

### Sound speeds

So far our partial expression for  $f$  contains no interaction term, no explicit coupling between the two fluid components. At first, this seems to imply that there are two sound modes, one in which the super fluid remains at rest, another where the normal component is quiet. This is, of course, very different from what is observed. The mistake is to overlook the imposition of a common temperature. Indeed we have seen, in Section VII.1, that the two modes are coupled, even in a mixture of two ideal gases, without an explicit interaction term in the free energy density.

In our treatment of mixtures of ideal gases or van der Waals fluids, to get a unique prediction of the two speeds, we asked that the unperturbed state satisfy the Gibbs-Dalton hypothesis; this has the effect of relating the two entropy parameters to each other. In superfluid Helium this is not a problem if  $S_2 = 0$  for the unperturbed state. The predicted speed matrix is thus unique, and so are the speeds as functions of the temperature and the

density. the result is that there are two modes, one in which the two densities are approximately in phase; another in which the phases are 180 degrees off.

As far as we understand, this is not the way that the speeds are calculated in Landau's theory. Instead one assumes (more or less correctly) that the phases are the same in one case and opposite in the other. It is perhaps only fair to ask for an estimate of the errors. A complete calculation, by Tilley and Tilley leads to a "small" internal contradiction. See Tilley and Tilley (1986), page 110. The finding that first sound is a density variation and that second sound is a temperature fluctuation is based on writing the equations of motion in the form (8.12) and the assumption that the last term is negligible for first sound and dominant for second sound. In this respect there is little difference between our model and Landau's theory. This observation tells us something interesting: The existence of 2 sounds in normal fluid (one of which is usually unobservable) and in the superfluid does not constitute a qualitative difference between superfluids and normal fluids, the distinction is only quantitative.

Guided by our previous experience with sound propagation in mixtures we would add a term  $\alpha\sqrt{\rho_1\rho_2}$  or  $\alpha\rho_1\rho_2$  to the free energy density, but let us make the more general assumption that the interaction takes the form

$$f_{\text{int}} = \alpha(\rho_1, \rho_2). \quad (8.2.6)$$

For a small deviation from an equilibrium mode we have the following equations

$$\frac{\ddot{\rho}_i}{\rho_i} = -\dot{v}_i = \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial \rho_i} (f + sT),$$

where  $f$  is the sum of the three terms (8.2.2), (8.2.4) and (8.2.6),  $\partial/\partial\rho_i$  are adiabatic derivatives - at constant  $S_1, S_2$ . Ignoring an additive constant,

$$\frac{\partial}{\partial \rho_1} (f + sT) = a(T) + c^2 \ln \rho_1 + S_1 T + \alpha_1, \quad (8.2.5)$$

$$\frac{\partial}{\partial \rho_2} (f + sT) = c^2 \ln \rho_2 + \alpha_2, \quad (8.2.6)$$

where suffixes on  $\alpha$  stand for derivatives. The adiabatic condition (8.2.4) makes  $T$  constant and the  $T$  dependent terms do not contribute. Hence

$$\frac{\ddot{\rho}_1}{\rho_1} = c^2 \frac{\rho_1''}{\rho_1} + \rho_1 \alpha_{11} \frac{\rho_1''}{\rho_1} + \rho_2 \alpha_{12} \frac{\rho_2''}{\rho_2},$$

$$\frac{\ddot{\rho}_2}{\rho_1} = c^2 \frac{\rho_2''}{\rho_2} + \rho_2 \alpha_{22} \frac{\rho_2''}{\rho_2} + \rho_1 \alpha_{21} \frac{\rho_1''}{\rho_1}.$$

The squared-speed matrix is

$$c^2 + \begin{pmatrix} \rho_1 \alpha_{11} & \rho_2 \alpha_{12} \\ \rho_1 \alpha_{21} & \rho_2 \alpha_{22} \end{pmatrix}.$$

The determinant of the matrix (without the  $c^2$  term) is

$$\rho_1 \rho_2 \det(\alpha_{ij}).$$

The case that the function  $\alpha$  is everywhere singular; that is, when this determinant vanishes identically, is especially easy to analyze, for in that case the eigenvalues are

$$c_1^2 = c^2, \quad c_2^2 = c^2 + \rho_1 \alpha_{11} + \rho_2 \alpha_{22}.$$

If  $\alpha$  is zero either component moves independently, with the same speed. If instead we had replaced the pressure potential  $c^2(\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2)$  by the expression favored by Arp and others, namely  $c^2 \rho \ln \rho$ , then only the total density would oscillate; that is, one of the speeds would be zero.

For a simple example, consider

$$\alpha(\rho_1, \rho_2) = \beta \rho \ln \rho.$$

Then the speed-squared matrix is

$$c^2 + \frac{\beta}{\rho} \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_1 & \rho_2 \end{pmatrix}.$$

The eigenvectors are:  $(1, 1)$  with eigenvalue  $c_1^2 = c^2 + \beta$  and  $(-\rho_2, \rho_1)$  with eigenvalue  $c_2^2 = c^2$ . The first is ordinary sound, both components oscillating in phase, the other is second sound with the components in opposite phase.

With a slight change of notation, the complete expression for  $f$  takes the form

$$f = A + B, \quad A = a(T)\rho_1, \quad (8.2.8)$$

$$B := (c_1^2 - c_2^2)\rho \ln \rho + c_2^2(\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2) + C(\rho_1 - \rho_2),$$

where  $c_1, c_2$  are the constant speeds of the two kinds of sound in the model. Other forms of interaction will give us speeds that vary with the concentration. The pressure is

$$p = c_1^2 \rho.$$

The last term in  $B$  makes no contribution to the pressure or to the speed of sound.

The result does not validate the approach, for the interaction turns out to overwhelm the the partial free energies, contradicting the evidence that the interaction between the two fluids is weak.

### Equilibrium

As was seen, existence of a unique equilibrium concentration for a given value of the temperature is complicated.

This is a state of rest, with both velocities equal to zero. It is determined by equalizing the chemical potentials

$$\mu_i = \left. \frac{\partial}{\partial \rho_i} (f + sT) \right|_{\rho, T}. \quad (8.2.9)$$

The relation that determines the concentration at equilibrium is  $\mu_1 = \mu_2$  or

$$\tilde{a}(T) + c_2^2 \ln \frac{\rho_1}{\rho_2} + 2C = 0, \quad (8.2.10)$$

where the function  $\tilde{a}$  is

$$\tilde{a}(T) = T \frac{da}{dT} - a. \quad (8.2.11)$$

The model is thus characterized by a fixed relationship between concentration and temperature, independent of  $\rho$  and  $p$ .

Can this be justified? In order to explain other experiment we have admitted that there are two independent densities and two independent vector fields, and that equilibrium is established solely by scattering off the wall. But Eq.(8.2.9) is the criterion that was used to determine the equilibrium of a mixture of Hydrogen atoms and hydrogen molecules, two gases that transform freely from one to the other. If we admit that the superfluid “mixture” has this property then it becomes difficult to understand the experiments that were discussed at the beginning of this chapter.

In liquid Helium below the  $\lambda$  point the concentration is often quoted as a fixed function of the temperature. In the model this relationship between concentration and temperature is determined by the choice that we make of the function  $a(T)$ , and we may choose this function so as to further enhance the similarity of the model to He II.

The experimental temperature profile for Helium shown in Table 8.1.2 is approximately

$$\frac{\rho_1}{\rho} = \left(\frac{T}{2.19}\right)^6, \quad T < 2.19. \quad (8.2.12)$$

Thus, for our model to have similar properties we should take, for  $0 < T < 2.19$ ,

$$\tilde{a}(T) = -2C - c_2^2 \ln \left( \left(\frac{T}{2.19}\right)^{-6} - 1 \right) \quad (8.2.13)$$

### The heat capacity

In the case of a heterogeneous system without chemical reactions or phase changes, what is measurable and usually quoted as  $C_V$  is

$$C_{\rho_1, \rho_2} = C_{\rho, x} = \left. \frac{\partial}{\partial T} U(S, T) \right|_{\rho_1, \rho_2},$$

where  $U$  is the specific internal energy density. It is measurable and well defined because it is possible to raise the temperature without affecting the values of the two densities.

But in the case of  $\text{He}_{\text{II}}$  and other systems with two mixed phases it is not possible to keep the concentration fixed. Heating takes the system through a sequence of equilibrium states in which the concentration is determined by  $\rho$  and  $T$ . What can be defined, and measured, is

$$C_{\rho_1, \rho_2} = C_{\rho, x} = \left. \frac{\partial}{\partial T} U(S, T) \right|_{\rho},$$

The full expression for the free energy density of the model is (8.2.8). The internal energy density, expressed in terms of the variables  $\rho_1, \rho_2$  and  $T$ ,

$$u(s, T) = \tilde{a}(T)\rho_1 - B$$

and the specific internal energy density is

$$U\left(\rho, \frac{\rho_1}{\rho_2}, T\right) = \tilde{a}(T)\frac{\rho_1}{\rho} - \frac{B}{\rho},$$

The specific heat at constant volume is

$$C_V = \left. \frac{d}{dT} U(\rho, \rho_1, T) \right|_{\rho}, \quad \frac{\rho_1}{\rho} = \left(\frac{T}{2.19}\right)^6.$$

The relevant total derivative is

$$\frac{d}{dT} = \frac{\partial \rho_1}{\partial T} \frac{\partial}{\partial \rho_1} + \frac{\partial}{\partial T}$$

and

$$C_V = \frac{\partial \tilde{a}}{\partial T} \frac{\rho_1}{\rho} + \frac{\tilde{a}}{\rho} \frac{\partial \rho_1}{\partial T} - \frac{1}{\rho} \frac{\partial B}{\partial \rho_1} \frac{\partial \rho_1}{\partial T}$$

at equilibrium the second and third terms cancel and

$$C_V = \frac{6c_2^2}{t} \frac{1}{t^{-6} - 1}, \quad t := \frac{T}{2.19}.$$

This is in close agreement with the observed specific heat for Helium II for temperatures below the  $\lambda$  point, if  $c_2^2 = 1/10$ . (In Joules/mol.)

### 8.3. Dynamical equations of the model

Since Landau's theory can be presented as a collection of formulas, let us list the equations of our model. All are derived from the Lagrangian density

$$\mathcal{L} = \sum_{i=1,2} \left( \rho_i (\dot{\Phi}_2 - \vec{\nabla} \Phi^2 / 2) \right) - f - sT,$$

with

$$f = a(T)\rho_1 + B(\rho_1\rho_2)s\rho_1S_1T.$$

- Two equations of continuity, derived from variation of the action with respect to the velocity potentials.

- An adiabatic condition, from variation of  $T$ , one of the fundamental relations of thermodynamics. In the simplest model it fixes the temperature in terms of the specific entropies.

- Variation of the densities gives the Bernoulli equations

$$-\frac{D\vec{v}_1}{Dt} = \text{grad}\left(\frac{\partial f}{\partial \rho_1} + S_1T\right) = 0, \quad -\frac{D\vec{v}_2}{Dt} = \text{grad}\frac{\partial f}{\partial \rho_2}.$$

Finally, at equilibrium,

$$\frac{\partial f}{\partial \rho_1} + S_1T = \frac{\partial f}{\partial \rho_2} + S_2T;$$

this from variation of the densities holding  $\rho$  fixed. That is all.

### The adiabatic index

We have not referred to the interpretation of this system as a manifestation of **Bose-Einstein** condensation, for it would take us far from the field in which we are comfortable. We are tempted, however, to speculate about what it is that characterizes, from our perspective, a system where a phase transformation makes part of the fluid move to a state in which the entropy is very near to zero. In Bose - Einstein condensation the lowest level in the atomic spectrum gets a high population and this appears to have an affect that resembles that of an energy gap.

This concept, that relates the properties of a liquid to interactions between the atoms, as well as interactions between the liquid and the walls, is a basic but difficult mainstay of kinetic theory. One of the important examples is the attempt to predict the value of the adiabatic index. The underlying idea is a principle of equipartition of energies. It suggests that  $2n$ , twice the adiabatic index, is the number of degrees of freedom of each atom. This idea needs some development and phenomenological adjustments; for example, vibrational degrees of freedom should not be counted. The idea encountered much resistance before being finally accepted.

To explain the fountain experiment it was suggested (London 1938) that the liquid that gets through the narrow channels carries no entropy. Entropy is related to the number of degrees of freedom but the adiabatic index is more directly so. While the entropy is a variable, not exactly zero and not constant, the adiabatic index of an ideal gas is a fixed parameter. For this reason we suggest that what characterizes the condensate, from the point of view of thermodynamics, is the adiabatic index, and that it is close to zero. This implies a low viscosity and it has an interesting effect on the propagation of sound in the mixture.

As we have seen, the existence of two modes of sound propagation is not unique to Helium, but a basic property of mixtures, although interactions between the components tend to damp one of the two. But in the case of this condensate model for superconducting Helium, the interaction would be weak and it makes sense to calculate the two velocities under the assumption that it is nearly negligible.

The speed of sound at 1 atmosphere and 293  $K$  is 985  $m/s$ , at the  $\lambda$  point it is 133.3  $K$ . It is well approximated in the interval by the formula of Laplace; a slightly better fit is achieved with the van der Waals equation of state, with  $n = 3/2$ .

We have done all necessary preparation for this in Section 7.2. If liquid helium is a mixture of two ideal gases then the speeds  $v_1, v_2$  are obtained as

$$v_i = c_i \sqrt{\mathcal{R}_1 T}, \quad i = 1, 2,$$

where  $c = c_1, c_2$  are the zeros of the determinant of the matrix

$$M(c) = \begin{pmatrix} 1 + \frac{\tau}{A} - c^2 & \frac{\kappa}{A} \\ \frac{\kappa\tau}{A} & \kappa + \frac{\kappa^2}{A} - c^2 \end{pmatrix},$$

with

$$A = n_1\tau + n_2\kappa, \\ \kappa = \frac{\mathcal{R}_2}{\mathcal{R}_1}, \quad \tau = \frac{\rho_1}{\rho_2}.$$

Below the critical point we use these formulas with  $m_1 = m_2$ ,  $\kappa = 1$ . The adiabatic index of the normal component is  $n_1 = 3/2$ , that of the superfluid component is denoted  $n$ , a free parameter. In this case the zeros of the matrix (7.1.11) are at

$$c^2 = 1 + \frac{\tau + 1}{n_1\tau + n} \quad \text{and at} \quad c^2 = 1.$$

The first value corresponds to the eigenvector  $\delta\rho_1 = \delta\rho_2$ ; the two densities oscillate in phase. In the other case the eigenvector satisfies  $\rho_1\delta\rho_1 = -\rho_2\delta\rho_2$ , the densities are in opposite phases. For an ideal gas this means that the pressure does not participate, as is characteristic of second sound in Helium II; it is an oscillation of the temperature only.

At equilibrium the ratio

$$x = \rho_1/(\rho_1 + \rho_2) = \frac{1}{1 + 1/\tau}$$

is related to the temperature,

$$x \approx \left(\frac{T}{2.19}\right)^6.$$

This is an experimental fact, and not applicable to an ideal gas. Nevertheless, let us make the naive assumption that the model can be tweaked to satisfy this relation. This is not completely unreasonable, since the formula for the total entropy should yield the value zero at  $T = 0$ . On this assumption we plot, in Fig. 8.1.5, both sounds below the  $\lambda$ -point,

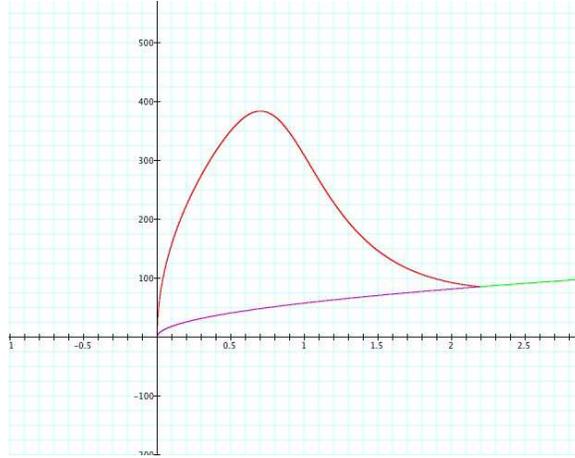


Fig. 8.3.1. The two speeds in Helium II, treated as a mixture of two ideal gases, a normal component with adiabatic index  $3/2$  and a superfluid component with adiabatic index  $1/100$ . the agreement with experiment is significant for  $T > 1$ .

namely

$$v_1 = \sqrt{5/3\mathcal{R}T} \sqrt{1 + \frac{\tau + 1}{3\tau/2 + n}}, \quad v_2 = \sqrt{5/3\mathcal{R}T}.$$

With diminishing value of the adiabatic parameter  $n$  of the superfluid, the sound profiles develop a greater similarity with those of superfluid Helium. A best fit in the interval  $1K < T < 2.19K$  is obtained for  $n = 1/100$ . With the usual kinetic interpretation of the adiabatic index we believe that this can be interpreted as representing a fluid with a very small space of excitations as well as a very weak interaction with the walls, thus providing a tentative explanation of the fountain effect.

### Rotational flow

A complete description of superfluids must deal with “rotons”, rotational flow, beyond the reach of Fetter-Walecka theory. We shall see, in Chapter XIV, that the required generalization involves two kinds of flows. The extension of this theory that is needed to deal with mixtures, and the correct approach to superfields, are far from clear.

*Project 5. Try to understand flux quantization.*

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